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# The Solution of Multiplicative Non-Homogeneous Linear Differential Equations

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## Abstract

In this study, taking particular solutions of non-homogeneous differential equations with constant coefficients in classical analysis as a basis, particular solutions of non-homogeneous differential equations in multiplicative analysis are obtained by using three methods namely operator method, the method of undetermined exponentials and the method of variation of parameters with constant exponentials.

**MSC:** 34A99

## Keywords

Multiplicative non-homogeneous linear differential equations, multiplicative derivative, the operator method, the method of undetermined exponentials, the method of variation of parameters with constant exponentials.

## 1. Introduction

Michael Grossman and Robert Katz indicated in their work [1] that infinitely many calculi can be generated independently. They also introduced a few of them namely geometric, anageometric, biogeometric and etc. This calculus also called non-Newtonian calculus. In non-Newtonian calculus, differentiation and integration are based on non-Newtonian operations instead of classical operations.

Geometric calculus, which depends on the division and multiplication operations instead of subtraction and addition operations for calculating differential and integral, was later named as multiplicative calculus by D. Stanley [2]. Then, some study with regard to multiplicative calculus is given by D. Campell [3]. The theoretical background of multiplicative calculus was given by Bashirov et al in [4]. Recent studies on multiplicative analysis [5-11] showed that some science and engineering problems can be solved in a more practical way by using this analysis.

This paper presents the particular solution for nonhomogeneous linear multiplicative differential equations with constant exponentials by using three methods namely operator method, the method of undetermined exponentials and the method of variation of parameters with constant exponentials.

## 2. Multiplicative Derivatives and Multiplicative Integrals

Here, we will give some basic definitions and properties of the multiplicative derivative theory which can be found in [2–5].

**Definition 2.1.** Let $f: R \to R^+$ be a positive function. The *multiplicative derivative* of the function $f$ is given by

$$\frac{d^*f}{dt}(t) = f^*(t) = \lim_{h\to 0}\left(\frac{f(t+h)}{f(t)}\right)^{\frac{1}{h}}. \quad (1)$$

Assuming that $f$ is a positive function and using properties of the classical derivative, multiplicative derivative can be written as

$$\frac{d^*f}{dt}(t) = f^*(t) = e^{\frac{f'(t)}{f(t)}} = e^{(\ln\circ f)'(t)} \tag{2}$$

for $(\ln\circ f)(t) = ln\big(f(t)\big)$ .

**Theorem 2.1.** Let $f$ and $g$ be differentiable with the multiplicative derivative. If $c$ be arbitrary constant, then $c.f,\ f.g,\ f+g,\ f/g\ ,\ f^g$ functions are differentiable with the multiplicative derivative and their multiplicative derivatives can be shown as

a) $(c.f)^*(t) = f^*(t)$,

b) $(f.g)^*(t) = f^*(t).g^*(t)$,

c) $(f+g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}} g^*(t)^{\frac{g(t)}{f(t)+g(t)}}$,

(3)

d) $(f/g)^* = f^*(t)/g^*(t)$,

e) $(f^g)^*(t) = f^*(t)^{g(t)} f(t)^{g'(t)}$.

**Definition 2.2.** A multiplicative integral is also defined in [4] for positive bounded functions and if $f$ is Riemann integrable on $[a,b]$, then

$$\int_a^b f(t)^{dt} = exp\left(\int_a^b (\ln f(t))dt\right) = e^{\int_a^b (\ln f(t))dt}. \tag{4}$$

This multiplicative integral has the properties:

a. $\int_a^b (f(t)^k)^{dt} = \left(\int_a^b \big(f(t)\big)^{dt}\right)^k,\ k\in\mathbb{R}$,

b. $\int_a^b \big(f(t)g(t)\big)^{dt} = \int_a^b \big(f(t)\big)^{dt} \int_a^b \big(g(t)\big)^{dt}$,

(5)

c. $\int_a^b \left(\frac{f(t)}{g(t)}\right)^{dt} = \frac{\int_a^b (f(t))^{dt}}{\int_a^b (g(t))^{dt}}$ ,

d. $\int_a^b f(t)^{dt} = \int_a^c f(t)^{dt} \int_c^b f(t)^{dt},\ a\le c\le\ b$

where $f$ and $g$ are multiplicative integrable on $[a,b]$.

## 3. Multiplicative Linear Differential Equations

**Definition 3.1.** Multiplicative linear differential equations can be defined in the form of

$$\left(y^{*(n)}\right)^{a_n(t)}\left(y^{*(n-1)}\right)^{a_{n-1}(t)} \dots (y^{**})^{a_2(t)}(y^*)^{a_1(t)} y^{a_0(t)} = f(t). \tag{6}$$

Here, $f(t)$ is a positive definite function.

If, all of $a_n(t)$ exponentials are constants, equation (6) is called as multiplicative linear differential equations with constant exponentials. Otherwise, equation (6) is as called multiplicative linear differential equations with variable exponentials. In equation (6), if $f(t) = 1$, equation (6) is called multiplicative homogeneous linear differential equations [11], otherwise it is called multiplicative nonhomogeneous linear differential equations.

**Definition 3.2.** Let $y_1, y_2, \dots, y_n$ be positive definite functions. Any expression of the form

$${y_1}^{c_1}{y_2}^{c_2} \dots {y_n}^{c_n} \tag{7}$$

is called a multiplicatively linear combinations of $y_1, y_2, \dots, y_n$ where $c_1, c_2, \dots, c_n$ are arbitrary constants.

**Definition 3.3.** Let $y_1, y_2, \dots, y_n$ be positive definite functions. Then, they are called multiplicatively linear dependent if there are not all zero constants $c_1, c_2, \dots, c_n$ with

$$y_1(t)^{c_1} y_2(t)^{c_2} \dots y_n(t)^{c_n} = 1 \tag{8}$$

for all $t$. Otherwise, they are called multiplicatively linear independent.

## 4. Multiplicative Non-homogeneous Linear Differential Equations with Constant Exponentials

Now, we analyze solutions of multiplicative non-homogeneous linear differential equation with constant exponentials. The most general linear multiplicative differential equations with constant exponentials is

$$\left(y^{*(n)}\right)^{a_n}\left(y^{*(n-1)}\right)^{a_{n-1}} \dots (y^{**})^{a_2}(y^{*})^{a_1}y^{a_0} = f(t), \tag{9}$$

where $y^{*(n)}(t) = e^{(\ln\circ y)^{(n)}(t)}$, $n$times, and the coefficients $a_0, a_1, \dots, a_{n-1}$ are real exponentials. The general solution of equation (9) is a product of a particular solution to the equation (9) and a general solution to the corresponding homogeneous equation. That is, if $y_h$ is general solution of the equation in homogeneous case [11] and $y_p$ is a particular solution to the equation (9), then general solution of the equation (9) is as;

$$y = y_h y_p.$$

In this work, we apply the operational method to derive a particular solution $y_p$ of (9).

**Lemma 4.1.** Let $L^*(\widetilde{D})$ is a linear operator with constant exponentials. Then,

$$L^*(\widetilde{D})e^{e^{rt}} = e^{L(r)e^{rt}}.$$

Where $r$ is real or complex constant and

$$L(r) = p_n r^n + p_{n-1}r^{n-1} + \dots + p_0.$$

This $L(r)$ equation is called as characteristic equation of equation (9).

**Proof:** For $y = e^{e^{rt}}$, We get

$$\widetilde{D}y = e^{re^{rt}}$$
$$\widetilde{D}^2 y = e^{r^2 e^{rt}}$$
$$\vdots$$
$$\widetilde{D}^n y = e^{r^n e^{rt}}.$$

Substituting $y = e^{e^{rt}}$ and its multiplicative derivative in case homogeneous of equation (9), then

$$L^*(\widetilde{D})\left(e^{e^{rt}}\right) = \left(e^{e^{rt}}\right)^{L(r)} \tag{10}$$

is obtained. Hence, the proof is completed.

### 4.1 The Operator Method

The multiplicative operator $\widetilde{D}(.)$ is defined as following [1]:

$$\widetilde{D}(.) = \frac{d^*(.)}{dt}.$$

Inverse of operator $\widetilde{D}$ is shown as $\widetilde{D}^{-1}$ . In this case, $(\widetilde{D}\widetilde{D}^{-1})y = y$. Because of we obtained $\widetilde{D}^{-1} = \tilde{I}$ , that is, operator $\tilde{I}$ is multiplicative integral operator. Similarly, for $L^*(\widetilde{D})y = f(t)$ multiplicative differential equation, it is written that $\left(\frac{1}{L^*(\widetilde{D})}L^*(\widetilde{D})\right)y = y$.Hence, if$L^*(\widetilde{D}) = \left(\widetilde{D}^n\right)^{a_n}\left(\widetilde{D}^{n-1}\right)^{a_{n-1}} \dots \left(\widetilde{D}^2\right)^{a_2}\left(\widetilde{D}\right)^{a_1}\left(\widetilde{D}^0\right)^{a_0}$, where $a_0, a_1, \dots, a_n$ are real exponentials, then

$$y = \frac{1}{L^*(\widetilde{D})}f(t).$$

**Theorem 4.1.1.** Let $L^*(\widetilde{D})$ be constant exponentials multiplicative operator and $L(D)$ be constant coefficients the classical operator. In this case, particular solution of the equation (9) is

$$y_p = \frac{1}{L^*(\widetilde{D})}f = e^{\frac{1}{L(D)}\ln f}.$$

**Proof:** For $L^*(\widetilde{D})y_p = f(t)$, we can write

$$y_p = \frac{1}{L^*(\widetilde{D})}f(t). \tag{11}$$

Let $\frac{1}{L^*(\widetilde{D})} f(t) = u(t)$. Appling linear multiplicative operator to both sides, we get

$$f(t) = L^*(\widetilde{D}) u(t).$$

From property of multiplicative linear operator, we have $L^*(\widetilde{D}) u(t) = e^{L(D)\ln u}$. Hence we write $f(t) = e^{L(D)\ln u}$ and $\ln f(t) = L(D)\ln u$. Hence, we get

$$u = e^{\frac{1}{L(D)}\ln f(t)}.$$

Substituting the last equation in (11), we obtain

$$y_p = \frac{1}{L^*(\widetilde{D})} f(t) = e^{\frac{1}{L(D)}\ln f}.$$

**Theorem 4.1.2.** If the characteristic equation $p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0 = 0$ of equation (9) have roots $r_1, r_2, \ldots, r_n$, then particular solution $y_p$ of equation (9) is

$$y_p = e^{e^{r_1 t}\int e^{(r_2 - r_1)t}\int e^{(r_3 - r_2)t} \ldots \int e^{(r_n - r_{n-1})t}\int e^{-r_n t}\ln f(t)(dt)^n}.$$

**Proof:** Because the characteristic equation $p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0 = 0$ of equation (9) have roots $r_1, r_2, \ldots, r_n$, equation (9) rewritten as following:

$$L^*(\widetilde{D}) y_p = e^{L(D)\ln y_p} = e^{[(D-r_1)(D-r_2)\ldots(D-r_n)]\ln y_p} = f(t).$$

Hence we can write as

$$[(D - r_1)(D - r_2)\ldots(D - r_{n-1})]\ln y_p = \frac{1}{(D-r_n)}\ln f(t).$$

Because of $\frac{1}{(D-r_n)}\ln f(t) = e^{r_n t} D^{-1}(e^{-r_n t}\ln f(t))$, we obtain

$$[(D - r_1)(D - r_2)\ldots(D - r_{n-1})]\ln y_p = e^{r_n t} D^{-1}(e^{-r_n t}\ln f(t)).$$

By continuing in this way;

$$\ln y_p = e^{r_1 t} D^{-1}\left(e^{(r_2 - r_1)t} \ldots D^{-1}\left(e^{(r_n - r_{n-1})t} D^{-1}(e^{-r_n t}\ln f(t))\right)\right)$$

is obtain. For $D^{-1} = I$, we have

$$y_p = e^{e^{r_1 t}\int e^{(r_2 - r_1)t}\int e^{(r_3 - r_2)t} \ldots \int e^{(r_n - r_{n-1})t}\int e^{-r_n t}\ln f(t)(dt)^n}.$$

Hence proof is completed.

**Theorem 4.1.3.** If the characteristic equation $p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0 = 0$ of equation (9) have distinct n-roots $r_1, r_2, \ldots, r_n$, then particular solution $y_p$ of equation (9) is

$$y_p = \prod_{k=1}^{n} e^{N_k e^{r_k t}\int \ln f(t) e^{-r_k t}(dt)}.$$

Where $N_1, N_2, \ldots, N_n$ provide equation $\frac{1}{L(D)} = \frac{N_1}{D-r_1} + \frac{N_2}{D-r_2} + \cdots + \frac{N_n}{D-r_n}$.

**Proof:** Because the characteristic equation $p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0 = 0$ of equation (9) have distinct n-roots $r_1, r_2, \ldots, r_n$, equation (9) rewritten as following:

$$L^*(\widetilde{D}) y_p = e^{L(D)\ln y_p} = e^{[(D-r_1)(D-r_2)\ldots(D-r_n)]\ln y_p} = e^{\ln f(t)}.$$

Hence particular solution $y_p$ is

$$y_p = e^{\frac{1}{[(D-r_1)(D-r_2)\ldots(D-r_n)]}\ln f(t)}.$$

If we choose $N_1, N_2, \ldots, N_n$ as $\frac{1}{[(D-r_1)(D-r_2)\ldots(D-r_n)]} = \frac{N_1}{D-r_1} + \frac{N_2}{D-r_2} + \cdots + \frac{N_n}{D-r_n}$, then we obtain

$$y_p = \prod_{k=1}^{n} e^{\frac{N_k}{D-r_k}\ln f(t)}.$$

Because of $\frac{N_1}{D-r_1}\ln f(t) = N_1 e^{r_1 t} I(\ln f(t)\, e^{-r_1 t})$, $\frac{N_2}{D-r_2}\ln f(t) = N_2 e^{r_2 t} I(\ln f(t)\, e^{-r_2 t})$, …, $\frac{N_n}{D-r_n}\ln f(t) = N_n e^{r_n t} I(\ln f(t)\, e^{-r_n t})$, the proof is completed.

**Theorem 4.1.4.** Let $L^*(\widetilde{D})$ is a linear operator with constant exponentials.

**(1)** If $f(t) = e^{e^{rt}}$ and $L(r) \neq 0$, then particular solution $y_p$ of (9) is

$$y_p = e^{\frac{1}{L(r)}e^{rt}}.$$

**(2)** If $f(t) = e^{b_n t^p + b_{n-1} t^{p-1} + \cdots + b_1 t + b_0}$, then particular solution $y_p$ of (9) is

$$y_p = e^{\frac{1}{L(D)} \ln f(t)} = e^{\frac{1}{D^k}[1 \pm \emptyset(D) \pm [\emptyset(D)]^2 \pm [\emptyset(D)]^3 \pm \cdots] \ln f(t)},$$

where $\frac{1}{L(D)} = \frac{1}{(D)^k} \frac{1}{1 \pm \phi(D)}$ .

**(3)** If $f(t) = e^{g(t)e^{rt}}$, then particular solution $y_p$ of (9) is

$$y_p = e^{\frac{1}{L(D)}e^{rt} g(t)} = e^{e^{rt}} e^{\frac{1}{L(D+r)} g(t)},$$

Where $r$ is a real or complex constant.

**(4)** If $L(r) = 0$ for the function of $f(t) = e^{e^{rt}}$, then particular solution $y_p$ of (1) is

$$y_p = e^{\frac{1}{L(D)}e^{rt}} = e^{\frac{e^{rt}}{\psi(r)} \cdot \frac{t^k}{k!}},$$

where $L(D) = (D - r)^k \psi(D)$.

**(5)** Let $L(D^2)$ is a linear operation with constant coefficients of $(D)^2$. If $L[-(r^2)] \neq 0$ for $f(t) = \begin{cases} e^{\sin rt} \\ e^{\cos rt} \end{cases}$, then particular solution $y_p$ of (1) is

$$y_p = e^{\frac{1}{L(D^2)} \begin{Bmatrix} \sin rt \\ \cos rt \end{Bmatrix}} = e^{\frac{1}{L(-r^2)} \begin{Bmatrix} \sin rt \\ \cos rt \end{Bmatrix}},$$

where $r$ is a real or complex constant.

**(6)** If $L[-(r^2)] = 0$ at for the functions of $f(t) = \begin{cases} e^{\sin rt} \\ e^{\cos rt} \end{cases}$ or if $f(t) = e^{t^p(\sin rt + \cos rt)}$, then the relations

$$cos\, rt = \frac{e^{irt} + e^{-irt}}{2}$$

$$sin\, rt = \frac{e^{irt} - e^{-irt}}{2i}$$

use to determine particular solution $y_p$ of (9). After, rule (3) apply it.

**Proof:** Each rule of theorem 4.3 can prove as in classical calculus. Here, we prove only the rule (1) and the rule (4).

**(1)** For $y_p$ is a solution, substituting $y_p$ in equation (1), we get

$$L^*(\widetilde{D}) y_p = e^{e^{rt}}$$

$$y_p = e^{\frac{1}{L(D)}e^{rt}} \tag{12}$$

From the classical calculus, we have

$$\frac{1}{L(D)} e^{rt} = \frac{1}{L(r)} e^{rt}$$

where $L(r) \neq 0$. If we substitute the last equation in equation (12), we obtain

$$y_p = e^{\frac{1}{L(r)}e^{rt}}.$$

**(4)**

$$y_p = \frac{1}{L^*(\widetilde{D})} e^{e^{rt}} = e^{\frac{1}{L(D)}e^{rt}} = e^{\frac{1}{(D-r)^k \psi(D)} e^{rt} .1}$$

$$y_p = e^{\frac{e^{rt}}{\psi(r)} \frac{1}{(D)^k} 1} = e^{\frac{e^{rt}}{\psi(r)} \frac{t^k}{k!}}$$

### 4.2 The Method of Undetermined Coefficients for Multiplicative Calculus

In this section, we apply the method of undetermined coefficients to derivate the particular solution of equation (9).

**1)** Let $f(t) = e^{Ae^{rt}}$. In this case, proper form for particular solution $y_p$ is

$$y_p = e^{\beta t^m e^{rt}},$$

where $m$ is the smallest non-negative integer which was selected that differentiated every terms in complementary function $y_h$, from every terms in particular solution $y_p$. Coefficient $\beta$ is found substituting multiplicative differential of $y_p$ and $y_p$ in equation (9).

**2)** Let $f(t) = e^{b_m t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + b_0}$. In this case, proper form for particular solution $y_p$ is

$$y_p = e^{A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0}.$$

$A_0, A_1, \ldots, A_m$ coefficients are found substituting multiplicative differential of $y_p$ and $y_p$ in equation (9). If 0 (zero) is $s$ times a root, then proper form for particular solution $y_p$ is

$$y_p = e^{t^s (A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0)}.$$

**3)** Let $f(t) = e^{e^{at}(b_m t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + b_0)}$. In this case, proper form for particular solution $y_p$ is

$$y_p = e^{t^s e^{rt} (A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0)},$$

where $s$ is the smallest non-negative integer which was selected that differentiable every terms in complementary function $y_h$, from every terms in particular solution $y_p$.

**4)** Let $f(t) = e^{e^{rt}(b_m t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + b_0)\left\{\begin{matrix} sin\alpha t \\ cos\beta t \end{matrix}\right\}}$. Then, proper form for particular solution $y_p$ is

$$y_p = e^{t^s [e^{rt}(A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0) cos\beta t + e^{rt}(B_m t^m + B_{m-1} t^{m-1} + \cdots + B_1 t + B_0) sin\alpha t]},$$

where $s$ is the smallest non-negative integer which was selected that differentiable every terms in complementary function $y_h$, from every terms in particular solution $y_p$.

### 4.3 Method of Variation of Parameters for Multiplicative Calculus

In this section, we apply method of variation of parameters to derivate the particular solution of equation (9).

**Theorem 4.3.1.** Let $y_h(t)$ be a function which is solution of homogenous case of equation (9) and given by

$$y_h(t) = \prod_{i=1}^{n} y_i(t)^{c_i} \quad . \tag{13}$$

Then particular solution of the equation (9) is

$$y_p(t) = \prod_{i=1}^{n} y_i(t)^{c_i(t)}$$

where $c_1(t), c_2(t), \ldots, c_n(t)$ provide following systems of equations

$$\begin{gathered} \prod_{i=1}^{n} y_i(t)^{c_i'} = 1 \\ \prod_{i=1}^{n} y_i^*(t)^{c_i'} = 1 \\ \vdots \\ \prod_{i=1}^{n} y_i^{*(n-2)}(t)^{c_i'} = 1 \\ \prod_{i=1}^{n} y_i^{*(n-1)}(t)^{c_i'} = f(t)^{\frac{1}{a_n}} \end{gathered} \tag{14}$$

**Proof:** Now we shall seek the solution of the equation (9) in the form

$$y_p(t) = \prod_{i=1}^{n} y_i(t)^{c_i(t)}.$$

If we calculate multiplicative derivative of $y_p(t)$, then we get

$$y_p^*(t) = \prod_{i=1}^{n}(y_i^*)^{c_i} \prod_{i=1}^{n}(y_i)^{c_i'}.$$

Let first condition is $\prod_{i=1}^{n}(y_i)^{c_i'} = 1$. In this case, we obtain

$$y_p^*(t) = \prod_{i=1}^{n}(y_i^*)^{c_i}.$$

If we calculate multiplicative derivative of $y_p^*(t)$, then we get

$$y_p^{**} = \prod_{i=1}^{n}(y_i^{**})^{c_i} \prod_{i=1}^{n}(y_i^*)^{c_i'}.$$

Let second condition is $\prod_{i=1}^{n}(y_i^*)^{c_i'} = 1$. So, we have

$$y_p^{**} = \prod_{i=1}^{n}(y_i^{**})^{c_i}.$$

By continuing in this way;

$$y_p^{*(n-1)} = \prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i} \prod_{i=1}^{n}\left(y_i^{*(n-2)}\right)^{c_i'}$$

is obtained. Let $(n-1)^{th}$condition is $\prod_{i=1}^{n}\left(y_i^{*(n-2)}\right)^{c_i'} = 1$. Hence, it is obtained that

$$y_p^{*(n-1)} = \prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i}.$$

And finally, calculating multiplicative derivative of above equation, we can write the following equation

$$y_p^{*(n)} = \prod_{i=1}^{n}\left(y_i^{*(n)}\right)^{c_i} \prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i'} .$$

If we substitute $y_p(t), y_p^*(t), y_p^{**} \dots, y_p^{*(n)}$ in the equation (9), then we have

$$\left(\prod_{i=1}^{n}\left(y_i^{*(n)}\right)^{c_i} \prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i'}\right)^{a_n} \left(\prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i}\right)^{a_{n-1}} \dots (\prod_{i=1}^{n}(y_i^*)^{c_i})^{a_1} (\prod_{i=1}^{n}(y_i)^{c_i})^{a_0} = f(t).$$

Because $y_1(t), y_2(t), \dots, y_n(t)$ are solutions of equation (13), we get

$$\left(\prod_{i=1}^{n}\left(y_i^{*(n)}\right)^{c_i}\right)^{a_n} \left(\prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i}\right)^{a_{n-1}} \dots (\prod_{i=1}^{n}(y_i^*)^{c_i})^{a_1} (\prod_{i=1}^{n}(y_i)^{c_i})^{a_0} = 1.$$

Hence,

$$\left(\prod_{i=1}^{n}\left(y_i^{*(n-1)}\right)^{c_i'}\right)^{a_n} = f(t)$$

is obtained. The last equation is $n^{th}$condition. The system of equations formed by these conditions is as (14)
If $c_1(t), c_2(t), \dots, c_n(t)$ provide system of equations which are built by conditions**,** then function $y_p(t) = \prod_{i=1}^{n} y_i(t)^{c_i(t)}$ is a particular solution of the equation (9).

## 5. Numerical Examples

Consider the following multiplicative differential equation

$$y^{**}(y^*)^3 y^2 = f(t).$$

Find particular solution for each of the following $f(t)$

**a.** $f(t) = e^{e^{2t}}$
**b.** $f(t) = e^{2t^2+t-3}$
**c.** $f(t) = e^{e^{2t}t}$

**Solution:**
**a)**

- *The operator method*:

The linear operator is $L(D) = D^2 + 3D + 2$. Since $L(2) = 12 \neq 0$, from Theorem 4.1.4 (1), the particular solution is obtained by

$$y_p = e^{\frac{1}{12}e^{2t}}.$$

- *The undetermined exponentials method:*

The proper form for particular solution $y_p$ is $y_p = e^{\beta t^m e^{2t}}$. If multiplicative differential of $y_p$ and $y_p$ in this equation substitute, then

$$e^{e^{2t}t^{m-2}[\beta m(m-1)]+e^{2t}t^{m-1}[7\beta m]+e^{2t}t^m[12\beta]} = e^{e^{2t}}$$

is obtained. Because the last equation provide for $m = 0$ that the smallest non-negative integer, we have $\beta = \frac{1}{12}$. Hence, the particular solution is

$$y_p = e^{\frac{1}{12}e^{2t}}.$$

- *The method of variation of parameters:*

The system of equations which are built by conditions is

$$\left(e^{e^{-2t}}\right)^{c_1{}'}\left(e^{e^{-t}}\right)^{c_2{}'} = 1$$
$$\left(e^{-2e^{-2t}}\right)^{c_1{}'}\left(e^{-e^{-t}}\right)^{c_2{}'} = e^{e^{2t}}.$$

Solving the above system of equations, we get $c_1{}' = -e^{4t}, c_2{}' = e^{3t}$. Hence, $c_1 = \frac{-1}{4}e^{4t}$ and $c_2 = \frac{1}{3}e^{3t}$.

The particular solution $y_p$ is

$$y_p = e^{\frac{1}{12}e^{2t}}.$$

**b)**

- *The operator method*:

$L^*(\widetilde{D})y_p = e^{L(D)\ln y_p} = e^{[D^2+3D+2]\ln y_p} = e^{2t^2+t-3}$. That is,

$$y_p = e^{\frac{1}{L(D)}\ln f(t)} = e^{\frac{1}{D^2+3D+2}(2t^2+t-3)}$$

Applying Theorem 4.1.4 (2), we obtain;

$$y_p = e^{\frac{1}{2\left(1+\frac{D^2+3D}{2}\right)}(2t^2+t-3)}$$
$$y_p = e^{\frac{1}{2}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2-\cdots\right](2t^2+t-3)}$$
$$y_p = e^{\frac{1}{2}\left[2t^2+t-3-\left(\frac{12t+7}{2}\right)+9\right]}$$
$$y_p = e^{t^2-\frac{5t}{2}+\frac{5}{4}}.$$

- *The undetermined exponentials method:*

The proper form for particular solution $y_p$ is $y_p = e^{A_2t^2+A_1t+A_0}$.If multiplicative differential of $y_p$ and $y_p$ in this equation substitute, then

$$e^{t^2(2A_2)+t(6A_2+2A_1)+2A_2+3A_1+2A_0} = e^{2t^2+t-3}$$

is obtained. By the last equation we get $A_2 = 1,\ A_1 = {}^{-5}/_{2}$ ve $A_0 = {}^{5}/_{4}$. If this equations substitute in $y_p$, we have

$$y_p = e^{t^2 - \frac{5t}{2} + \frac{5}{4}}.$$

- *The method of variation of parameters:*

The system of equations which are built by conditions is

$$\left(e^{e^{-2t}}\right)^{c_1{}'}\left(e^{e^{-t}}\right)^{c_2{}'} = 1$$

$$\left(e^{-2e^{-2t}}\right)^{c_1{}'}\left(e^{-e^{-t}}\right)^{c_2{}'} = e^{2t^2+t-3}$$

If we solve this system of equations, we obtain

$$c_1 = \frac{-1}{2} e^{2t}(2t^2 + t - 3) + \frac{4t-1}{4} e^{2t}$$

$$c_2 = e^{t}(2t^2 - 3t)$$

Hence, we obtain particular solution $y_p$ as following:

$$y_p = e^{t^2 - \frac{5t}{2} + \frac{5}{4}}.$$

**c)**

- *The operator method*:

$L^*(\widetilde{D})y_p = e^{L(D)\ln y_p} = e^{[D^2+3D+2]\ln y_p} = e^{e^{2t}t}$. That is,

$$y_p = e^{\frac{1}{D^2+3D+2}\left(e^{2t}t\right)}$$

Applying Theorem 4.1.4 (3), we obtain;

$$y_p = e^{e^{2t}\frac{1}{(D+2)^2+3(D+2)+2}t}$$

$$y_p = e^{e^{2t}\frac{1}{D^2+7D+12}t}$$

$$y_p = e^{e^{2t}\frac{1}{12\left(1+\frac{D^2+7D}{12}\right)}t}$$

$$y_p = e^{e^{2t}\frac{1}{12}\left[1-\frac{D^2+7D}{12}+\cdots\right]t}$$

$$y_p = e^{e^{2t}\left(\frac{t}{12}-\frac{7}{144}\right)}.$$

- *The undetermined exponentials method:*

The proper form for particular solution $y_p$ is $y_p = e^{t^s e^{2t(A_1 t + A_0)}}$.If multiplicative differential of $y_p$ and $y_p$ in this equation substitute, then

$$e^{e^{2t(t^{s+1})[12A_1]} + e^{2t(t^s)[12A_0+7A_1(s+1)]} + e^{2t(t^{s-1})[7A_0 s + A_1(s+1)s]} + e^{2t(t^{s-2})[A_0(s-1)s]}} = e^{e^{2t}t}$$

is obtained. Because the last equation provide for $s = 0$ that the smallest non-negative integer, we get $A_1 = {}^{1}/_{12}$ and $A_0 = {}^{-7}/_{144}$

$$y_p = e^{e^{2t}\left(\frac{t}{12}-\frac{7}{144}\right)}.$$

- *The method of variation of parameters:*

The system of equations which are built by conditions is

$$\left(e^{e^{-2t}}\right)^{c_1{}'}\left(e^{e^{-t}}\right)^{c_2{}'}=1$$

$$\left(e^{-2e^{-2t}}\right)^{c_1{}'}\left(e^{-e^{-t}}\right)^{c_2{}'}=e^{e^{2t}t}$$

If we solve this system of equations, we obtain $c_1=-\frac{1}{4}e^{4t}t+\frac{1}{16}e^{4t}$ and $c_2=\frac{1}{3}e^{3t}t-\frac{1}{9}e^{3t}$

Hence, we obtain particular solution $y_p$ as following:

$$y_p=e^{e^{2t}\left(\frac{t}{12}-\frac{7}{144}\right)}.$$

## 6. Conclusion

In this work, multiplicative linear differential equations are discussed and the theory of multiplicative non-homogeneous differential equations with constant exponentials is investigated. The concept of multiplicative differential operator and its properties are studied and we get the solutions of multiplicative non-homogeneous differential equations with constant exponentials with the help of this operator (the operator method).As well as operator method we also give the particular solution in non-homogeneous case via the method of undetermined exponentials and the method of variation of parameters. We support the findings obtained in this study with numerical examples. As shown by numerical examples, results are consistent with each of these three methods. Further, it is appeared that the results obtained in this work correspond to results which are obtained in ordinary case.

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## Competing interests

The authors declare that they have no competing interests regarding the publication of this manuscript.

## Authors' contributions

Both authors contributed equally and significantly in writing this manuscript. Both authors read and approved the final manuscript.

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