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Improved High Order Methods Using Boundary Layer Detection for a Singular Perturbation Problem

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Abstract

A singular perturbation problem is solved with improved high order methods using boundary layer detection theorems. The stability and convergence, independent of the singular perturbation parameter, is numerically verified.

Keywords

Singular Perturbation, Differential Equations, Boundary Layers

We consider the singular perturbation problem,

$$\begin{aligned} &\varepsilon u'' - uu' = 0 \quad x \in (-1,1), \\ &u(-1) = 0 \quad \text{and} \quad u(1) = -1. \end{aligned} \tag{1}$$

from Chang [1], O'Malley [2] and Miller [3]. Using the boundary layer detection theorem in Zhang [4], an exponential boundary layer of width proportional the singular perturbation parameter can be found to the left boundary. Because of the presence of the boundary layer, the numerical methods from Keller [5] will have to be adjusted accordingly. Choudhury [6] and Ilicasu [7] solved the problem using nonstandard high order methods on a uniform mesh. With the theory of weak formulation in Lax [8], Zhang [9] proposed test functions fitting the exponential boundary layer to solve the singular perturbation. In this paper, the singular perturbation problem is solved with improved high order methods based on the boundary layer detection theorem. Furthermore, a sixth order method is developed and verified numerically.

An improvement on the 4th order method

Let $\omega = \frac{1}{\varepsilon}$ where ε is the singular perturbation parameter in the problem (1). Then we rewrite the second order derivative and compute the third and fourth order derivatives,

$u'' = \omega uu'$,

$u''' = \omega (u')^2 + \omega uu''$, and

$$u^{(4)} = 2\omega u'u'' + \omega u'u'' + \omega u\, u''' = 3\omega u'u'' + \omega^2 u\left(u'\right)^2 + \omega^2 u^2 u'' = \omega^2 u\left(u'\right)^2 + \left(3\omega u' + \omega^2 u^2\right)u''$$

Setting $A_3 = \omega u_i', B_3 = \omega u_i$, and $A_4 = \omega^2 u_i u_i', B_4 = 3\omega u_i' + \omega^2 u_i^2$, we get $u_i^{(3)} = A_3 u_i' + B_3 u_i''$, $u_i^{(4)} = A_4 u_i' + B_4 u_i''$.

For the purpose of simplicity, we continue to use the following new notations,

$$A^{**} = \frac{h^4 A_4}{24},\ B^{**} = h + \frac{h^3 A_3}{6},\ C^{**} = \frac{h^2}{2} + \frac{h^4 B_4}{24},\ D^{**} = \frac{h^3 B_3}{6}$$

To improve the accuracy of the process, we develop the fourth order finite differences to approximate the singular perturbation problem (1).

$\varepsilon u_i'' - u_i u_i' \approx c_3^{**} u_{i+1} + c_2^{**} u_i + c_1^{**} u_{i-1}$ where

$$c_3^{**} = \frac{-u_i D^{**} - \varepsilon B^{**} + \varepsilon A^{**} + u_i C}{2\left(A^{**}D^{**} - B^{**}C^{**}\right)},$$

$$c_1^{**} = \frac{-u_i D^{**} - \varepsilon B^{**} - \left(\varepsilon A^{**} + u_i C^{**}\right)}{2\left(A^{**}D^{**} - B^{**}C^{**}\right)},$$

$$c_2^{**} = -\left(c_3^{**} + c_1^{**}\right) = \frac{u_i D^{**} + \varepsilon B^{**}}{A^{**}D^{**} - B^{**}C^{**}}.$$

In Ilicasu [7], the derivative u_i' in A_3, A_4 and B_4 is replaced with $u_i' = \dfrac{u_{i+1} - u_{i-1}}{2h}$. We now use the fourth order finite differences to establish a higher order approximation to the derivative u_i'. For $i = 1, 2, \ldots, N-1$, let the first derivative be, $u_i' = c_3 u_{i+1} + c_2 u_i + c_1 u_{i-1}$ where c_3, c_2 and c_1 are constants.

By Taylor series expansion, we have

$$\begin{aligned}
u_i' &= c_3\left(u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{2}u_i''' + \frac{h^7}{24}u_i^{(4)} + \ldots\right) \\
&\quad + c_2 u_i + c_1\left(u_i - h u_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i^{(4)} + \ldots\right) \\
&\approx c_3\left\{u_i + h u_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}\left(\omega u_i'^2 + \omega u_i u_i''\right) + \frac{h^4}{24}\left[\omega^2 u_i u_i'^2 + \left(3\omega u_i' + \omega^2 u_i^2\right)u_i''\right]\right\} \\
&\quad + c_2 u_i + c_1\left\{u_i - h u_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}\left(\omega u_i'^2 + \omega u_i u_i''\right) + \frac{h^4}{24}\left[\omega^2 u_i u_i'^2 + \left(3\omega u_i' + \omega^2 u_i^2\right)u_i''\right]\right\} \\
&= \left(c_3 + c_2 + c_1\right)u_i + \left[\left(c_3 - c_1\right)\left(h + \frac{h^3}{6}\omega u_i'\right) + \left(c_3 + c_1\right)\frac{h^4}{24}\omega^2 u_i u_i'\right]u_i' \\
&\quad + \left[\left(c_3 + c_1\right)\left(\frac{h^2}{2} + \frac{h^4}{24}3\omega u_i' + \frac{h^4}{24}\omega^2 u_i^2\right) + \left(c_3 - c_1\right)\frac{h^3}{6}\omega u_i\right]u_i''.
\end{aligned}$$

Matching the corresponding coefficients, we obtain the following system of equations

$$\begin{aligned}
&c_3 + c_2 + c_1 + 0, \\
&\left(c_3 - c_1\right)\left(h + \frac{h^3}{6}\omega u_i'\right) + \left(c_3 + c_1\right)\frac{h^4}{24}\omega^2 u_i u_i' = 1, \\
&\left(c_3 + c_1\right)\left(\frac{h^2}{2} + \frac{h^4}{24}3\omega u_i' + \frac{h^4}{24}\omega^2 u_i'^2\right) + \left(c_3 - c_1\right)\frac{h^3}{6}\omega u_i = 0.
\end{aligned}$$

To understand the system better, we create more notations,

$$
\begin{aligned}
A &= \frac{h^4}{24}\omega^2 u_i u_i', \\
B &= h + \frac{h^3}{6}\omega u_i', \\
C &= \frac{h^2}{2} + \frac{h^4}{24}3\omega u_i' + \frac{h^4}{24}\omega^2 u_i'^2, \\
D &= \frac{h^3}{6}\omega u_i,
\end{aligned}
$$

It is clear the system of equations is equivalent to

$$
\begin{aligned}
&c_3 + c_2 + c_1 = 0, \\
&(c_3 + c_1)A + (c_3 - c_1)B = 1, \\
&(c_3 + c_1)C + (c_3 - c_1)D = 0,
\end{aligned}
$$

of which, $c_3 + c_1$ and $c_3 - c_1$ are,

$$
\begin{aligned}
c_3 + c_1 &= \frac{D}{AD - BC}, \\
c_3 - c_1 &= \frac{-C}{AD - BC}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
c_3 &= \frac{D - C}{2(AD - BC)}, \\
c_1 &= \frac{D + C}{2(AD - BC)}, \\
c_2 &= -(c_3 + c_1) = \frac{-D}{AD - BC}.
\end{aligned}
$$

The error term is $\frac{h}{120}\left(c_3 u^{(5)}(\eta_1) + c_1 u^{(5)}(\eta_2)\right)$ where $\eta_1, \eta_2 \in [x_i - h, x_i + h]$.

Now c_3 and c_1 are updated to the fourth order accuracy. The improvement of the method is verified by numerical experiments.

The 6th Order Method

The second improvement is to add more terms from the Taylor series. We expand the u_{i+1} and u_{i-1} up to the sixth order derivatives. The following is a development of the sixth order method. The fifth order method is developed by dropping the sixth order derivative terms. We consider the fifth order and sixth order derivatives from the singular perturbation problem (1):

$$
\begin{aligned}
u^{(5)} &= 3\omega u''^2 + 3\omega u'u''' + \omega^2 u'u'^2 + 2\omega^2 u\, u'u'' + 2\omega^2 u\, u'u'' + \omega^2 u^2 u''' \\
&= 3\omega u''^2 + 3\omega u'\left(\omega u'^2 + \omega uu''\right) + \omega^2 u'^3 + 4\omega^2 uu'u'' + \omega^2 u^2\left(\omega u'^2 + \omega uu''\right) \\
&= 3\omega u''^2 + 3\omega^2 u'^3 + 3\omega^2 uu'u'' + \omega^2 u'^3 + 4\omega^2 uu'u'' + \omega^2 u^2 u'^2 + \omega^3 u^3 u'' \\
&= 3\omega u''^2 + 4\omega^2 u'^3 + 7\omega^2 uu'u'' + \omega^3 u^2 u'^2 + \omega^3 u^3 u'' \\
&= \left(4\omega^2 u'^2 + \omega^3 u^2 u'\right)u' + \left(3\omega u'' + 7\omega^2 uu' + \omega^3 u^3\right)u''
\end{aligned}
$$

and

$$u^{(6)} = 6\omega u''u''' + 12\omega^2 u'^2 u'' + 7\omega^2\left[\left(u'^2 + uu''\right)u'' + uu'u''\right]$$
$$+2\omega^3 u u'^3 + 2\omega^3 u^2 u'u'' + 3\omega^3 u^2 u'u'' + \omega^3 u^3 u'''$$
$$= 6\omega u''\left(\omega u'^2 + \omega uu''\right) + 12\omega^2 u'^2 u'' + 7\omega^2\left[u'^2 u'' + uu''^2 + uu'\left(\omega u'^2 + \omega uu''\right)\right]$$
$$+2\omega^3 uu'^3 + 5\omega^3 u^2 u'u'' + \omega^3 u^3\left(\omega u'^2 + \omega uu''\right)$$
$$= 6\omega^2 u'^2 u'' + 6\omega^2 uu''^2 + 12\omega^2 u'^2 u'' + 7\omega^2 u'^2 u'' + 7\omega^2 uu''^2 + 7\omega^3 uu'^3 + 7\omega^3 u^2 u'u''$$
$$+2\omega^3 u u'^3 + 5\omega^3 u^2 u'u'' + \omega^4 u^3 u'^2 + \omega^4 u^4 u''$$
$$= 25\omega^2 u'^2 u'' + 13\omega^2 uu''^2 + 7\omega^3 uu'^3 + 2\omega^3 uu'^3 + 12\omega^3 u^2 u'u'' + \omega^4 u^3 u'^2 + \omega^4 u^4 u''$$
$$= \left(9\omega^3 uu'^2 + \omega^4 u^3 u'\right)u' + \left(25\omega^2 u'^2 + 13\omega^2 uu'' + 12\omega^3 u^2 u' + \omega^4 u^4\right)u''.$$

For simplicity, we rewrite the derivatives

$u_i^{(3)} = A_3 u_i' + B_3 u_i''$ where $A_3 = \omega u_i', B_3 = \omega u_i$,

$u_i^{(4)} = A_4 u_i' + B_4 u_i''$ where $A_4 = \omega^2 u_i u_i', B_4 = 3\omega u_i' + \omega^2 u_i^2$,

$u_i^{(5)} = A_5 u_i' + B_5 u_i''$ where $A_5 = 4\omega^2 u_i'^2 + \omega^3 u_i^2 u_i', B_5 = 3\omega u_i'' + 7\omega^2 u_i u_i' + \omega^3 u_i^3$,

and

$u_i^{(6)} = A_6 u_i' + B_6 u_i''$ where $A_6 = 9\omega^3 u_i u_i'^2 + \omega^4 u_i^3 u_i'$, and

$B_6 = 25\omega^3 u_i'^2 + 13\omega^2 u_i u_i'' + 12\omega^3 u_i^2 u_i' + \omega^4 u_i^4$

We write

$\varepsilon u_i'' - u_i u_i' = c_3^* u_{i+1} + c_2^* u_i + c_1^* u_{i-1}$,

where c_3^*, c_2^* and c_1^* are constants. By Taylor series expansion, we obtain

$$\varepsilon u_i'' - u_i u_i' = c_3^* u_{i+1} + c_2^* u_i + c_1^* u_{i-1}$$
$$\approx c_3^*\left[u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i^{(4)} + \frac{h^2}{120}u_i^{(5)} + \frac{h^6}{720}u_i^{(6)}\right]$$
$$+c_2^* u_i + c_1^*\left[u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i^{(4)} - \frac{h^5}{120}u_i^{(5)} + \frac{h^6}{720}u_i^{(6)}\right]$$
$$= c_3^*\left[u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}\left(A_3 u_i' + B_3 u_i''\right) + \frac{h^2}{24}\left(A_4 u_i' + B_4 u_i''\right) + \frac{h^5}{120}\left(A_5 u_i' + B_5 u_i''\right) + \frac{h^6}{720}\left(A_6 u_i' + B_6 u_i''\right)\right]$$
$$+c_2^* u_i + c_1^*\left[u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}\left(A_3 u_i' + B_3 u_i''\right) + \frac{h^2}{24}\left(A_4 u_i' + B_4 u_i''\right) - \frac{h^5}{120}\left(A_5 u_i' + B_5 u_i''\right) + \frac{h^6}{720}\left(A_6 u_i' + B_6 u_i''\right)\right]$$
$$= c_3^*\left[u_i + \left(h + \frac{h^3 A_3}{6} + \frac{h^4 A_4}{24} + \frac{h^5 A_5}{120} + \frac{h^6 A_6}{720}\right)u_i' + \left(\frac{h^2}{2} + \frac{h^3 B_3}{6} + \frac{h^4 B_4}{24} + \frac{h^5 B_5}{120} + \frac{h^6 B_6}{720}\right)u_i''\right] + c_2^* u_i$$
$$+c_1^*\left[u_i + \left(-h - \frac{h^3 A_3}{6} + \frac{h^4 A_4}{24} - \frac{h^5 A_5}{120} + \frac{h^6 A_6}{720}\right)u_i' + \left(\frac{h^2}{2} - \frac{h^3 B_3}{6} + \frac{h^4 B_4}{24} - \frac{h^5 B_5}{120} + \frac{h^6 B_6}{720}\right)u_i''\right]$$
$$= \left(c_3^* + c_2^* + c_1^*\right)u_i + \left[\left(c_3^* + c_1^*\right)\left(\frac{h^4 A_4}{24} + \frac{h^6 A_6}{720}\right) + \left(c_3^* - c_1^*\right)\left(h + \frac{h^3 A_3}{6} + \frac{h^5 A_5}{120}\right)\right]u_i'$$
$$+\left[\left(c_3^* + c_1^*\right)\left(\frac{h^2}{2} + \frac{h^4 B_4}{24} + \frac{h^6 B_6}{720}\right) + \left(c_3^* - c_1^*\right)\left(\frac{h^3 B_3}{6} + \frac{h^5 B_5}{120}\right)\right]u_i''$$

Matching the corresponding coefficients of both sides, we get the following system of equation in terms of c_3^*, c_2^* and c_1^*:

$$c_3^* + c_2^* + c_1^* = 0,$$
$$\left(c_3^* + c_1^*\right) A^* + \left(c_3^* - c_1^*\right) B^* = -u_i,$$
$$\left(c_3^* + c_1^*\right) C^* + \left(c_3^* - c_1^*\right) D^* = \varepsilon,$$

where

$$A^* = \frac{h^4 A_4}{24} + \frac{h^6 A_6}{720}, B^* = h + \frac{h^3 A_3}{6} + \frac{h^5 A_5}{120}, C^* = \frac{h^2}{2} + \frac{h^4 B_4}{24} + \frac{h^6 B_6}{720}, D^* = \frac{h^3 B_3}{6} + \frac{h^5 B_5}{120}.$$

Note that the derivatives contained in A_3, A_4, A_5 and B_3, B_4, B_5 are replaced with the following:

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2h}, u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

Thus, we get

$$c_3^* + c_1^* = \frac{-u_i D^* - \varepsilon B^*}{A^* D^* - B^* C^*},$$
$$c_3^* - c_1^* = \frac{\varepsilon A^* + u_i C^*}{A^* D^* - B^* C^*}.$$

Therefore, the solution is

$$C_3^* = \frac{-u_i D^* - \varepsilon B^* + \varepsilon A^* + u_i C^*}{2\left(A^* D^* - B^* C^*\right)},$$

$$C_1^* = \frac{-u_i D^* - \varepsilon B^* - \left(\varepsilon A^* + u_i C^*\right)}{2\left(A^* D^* - B^* C^*\right)},$$

$$c_2^* = -\left(c_3^* + c_1^*\right) = \frac{u_i D^* + \varepsilon B^*}{A^* D^* - B^* C^*}.$$

The error term is $\frac{h^2}{5040}\left(c_3 u^{(7)}\left(\eta_3\right) + c_1 u^{(7)}\left(n_4\right)\right)$ where $\eta_3, \eta_4 \in \left[x_i - h, x_i + h\right]$.

The results are improved as shown in the tables. We compare the numerical results among different methods. For the methods of Choudhury [6] and Ilicasu [7], a uniform mesh is used with N=2,000 mesh points. For the improved 4th order, 5th order and 6th order methods, the number N_n of mesh points on the non-boundary layer domain is 170 and the number N_b of points on the boundary layer is 300. In addition to the improved accuracy, the computing cost is reduced significantly thanks to fewer number of mesh points.

Table 1. Maximal error comparison among different methods ε=0.01

Method	Number of Points	Number of Iterations	Max Error
Choudhury's Method [6]	2,000	Not known	$2.91*10^{-2}$
2nd order of Ilicasu [7]	2,000	3,201	$2.61*10^{-4}$
4th order of Ilicasu [7]	2,000	3,152	$1.00*10^{-5}$
Improved 4th order method of this paper	470	697	$8.40*10^{-5}$
5th order method of this paper	470	697	$1.34*10^{-7}$
6th order method of this paper	470	697	$7.71*10^{-8}$

For the methods of this paper, the tolerance of iteration is set at 10^{-10}.

The improved high order methods work well for much smaller values of the singular perturbation parameter. The convergence of the improved fourth order method is shown in Table 2, with the smallest values of singular perturbation parameter $\varepsilon = 10^{-12}$.

Table 2. The convergence of the improved fourth order method

Number of Points	Maximal Error		
	ε=10^{-5}	ε=10^{-10}	ε=10^{-12}
N=350 (N_n=200, N_b=150)	3.53*10^{-4}	3.53*10^{-4}	3.56*10^{-4}
N=400 (N_n=200, N_b=200)	1.91*10^{-4}	1.94*10^{-4}	1.94*10^{-4}
N=450 (N_n=200, N_b=250)	1.20*10^{-4}	1.22*10^{-4}	1.35*10^{-4}
N=500 (N_n=200, N_b=300)	8.39*10^{-5}	8.40*10^{-5}	9.71*10^{-5}

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