Advances in Computer and Communication, 2020, 1(1), 36-45
http://www.hillpublisher.com/journals/acc/
ISSN Online: ****-****
ISSN Print: ****-****

A Comparative Study of q-Homotopy Analysis Method and Liao's Optimal Homotopy Analysis Method

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How to cite this paper: Shaheed N. Huseen. (2020) A Comparative Study of q-Homotopy Analysis Method and Liao's Optimal Homotopy Analysis Method. *Advances in Computer and Communication*, 1(1), 36-45.
DOI: 10.26855/acc.2020.12.004

Received: October 30, 2020
Accepted: November 28, 2020
Published: December 15, 2020

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Abstract

In this paper, comparative study of q-homotopy analysis method (q-HAM) with the Liao's optimal homotopy analysis method (OHAM) is proposed. We solved two examples, first example is a system of Volterra integro-differential equations and the second one is a nonlinear integro-differential equation. The results show that the q-HAM was more accuracy than the OHAM.

Keywords

q-homotopy analysis method, Convergence parameter, Integro-differential equations

1. Introduction

A numerous methods were proposed to find approximate solutions for nonlinear phenomena of our life. Liao [1-5] proposed an analytic method for solving linear and nonlinear problems, namely homotopy analysis method (HAM). Recently, HAM has been successfully applied to solve different types of non-linear problems in variety fields [6, 7, 8, 9, 10]. HAM contains a convergence parameter h which with a simple way provides us to adjust and control the convergence region of the series solution. Moreover, by means of h-curve, easily to gain the valid regions of h to find a convergent series solution. We cannot determine the best value of h by plotting the h-curves, to find the fastest convergent series. Yabushitaet al. [11] suggested the "optimization method" to find out the two optimal values of h by the minimum of the squared residual error. Akyildiz and Vajravelu [12] determine an optimal value of h by the minimum of squared residual and found that the homotopy-series solution converges so quickly. A one-step optimal homotopy analysis method was proposed by Niu and Wang [13]. Liao [14] introduced an optimal HAM with three convergence parameters. A general method of HAM namely q-homotopy analysis method (q-HAM) was proposed by El-Tawil and Huseen [15], the q-HAM contains two convergence parameters n and h such that the case of $n = 1$ the HAM can be reached. Many researches applied the q-HAM to numerous problems in different felids [16-28].

2. Fundamental Idea ofq-HAM

Consider the following differential equation:

$$N[u(t)] = 0, \tag{1}$$

where N is a nonlinear operator, $u(t)$ is an unknown function. The zero-order deformation equation of q-HAM is

$$(1 - nq)L[\emptyset(t,q) - u_0(t)] = qhN[\emptyset(t,q)], \tag{2}$$

where ≥ 1 , $0 \leq q \leq \frac{1}{n}$ is the embedded parameter , L is a linear operator and $h \neq 0$.

If $q = 0$ and $q = \frac{1}{n}$ then equation (2) becomes

$$\emptyset(t,0) = u_0(t) \ , \qquad \emptyset\left(t,\frac{1}{n}\right) = u(t) \tag{3}$$

Respectively. Thus as q varies from 0 to $\frac{1}{n}$, the solution $\emptyset(t,q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$.

Taking the Taylor series of $\emptyset(t,q)$ we obtain

$$\emptyset(t,q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m \ , \tag{4}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t,q)}{\partial q^m} \ \Big|_{q=0} \tag{5}$$

Let we choose $h, u_0(t),\ L$ such that the series (4) converges at $q = \frac{1}{n}$ and

$$u(t) = \emptyset\left(t,\frac{1}{n}\right) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)\left(\frac{1}{n}\right)^m \tag{6}$$

Defining the vector $u_r(t) = \{u_0(t), u_1(t), u_2(t), \ldots, u_r(t)\}$, differentiating Eq. (2) m times with respect to q and then setting $q = 0$ and dividing by $m!$ we have the m^{th} order deformation equation

$$L[u_m(t) - k_m u_{m-1}(t)] = hR_m(\vec{u}_{m-1}(t)), \tag{7}$$

where

$$R_m(\vec{u}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\emptyset(t,q)]}{\partial q^{m-1}} \ \Big|_{q=0} \ , \tag{8}$$

and

$$k_m = \begin{cases} 0 & m \leq 1 \\ n & otherwise \end{cases} \tag{9}$$

Let

$$\Delta_m = \int (N[U_m(t)])^2 d\,\Omega \ , \quad t \in \Omega \ , \tag{10}$$

where

$$U_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t)\left(\frac{1}{n}\right)^k,$$

is the square residual error of the mth-order approximation of the Eq. (1) integrated in the whole domain Ω. In theory if the square residual error Δ_m tends to zero, then

$$U_m(t) = \sum_{k=0}^{m} u_k(t)\left(\frac{1}{n}\right)^k$$

is a series solution of the original equation (1). Besides, at the given order of approximation, the minimum of the squared residual error Δ_m corresponds to the optimal approximation. So, the curves of the squared residual error Δ_m versus h indicate to the valid region of h, hence the optimal value of h that corresponds to the minimum of Δ_m.

Liao [14] introduced the so-called average residual error $E_m = \frac{1}{K}\sum_{j=0}^{K}[N(\sum_{k=0}^{m} u_k(j\Delta x))]^2$ where $\Delta x = \frac{1}{K}$.

3. Applications

Example 3.1. Consider the following system of Volterra integro-differential equations [29]

$$u_1' = 1 + t + t^2 - u_2(t) - \int_0^t \big(u_1(s) + u_2(s)\big)ds \ , \qquad u_1(0) = 1, \tag{11}$$

$$u_2' = -1 - t + u_1(t) - \int_0^t \big(u_1(s) - u_2(s)\big)ds\,, \qquad u_2(0) = -1,$$

With exact solutions $u_1(t) = t + \exp(t)$ and $u_2(t) = t - \exp(t)$.

Let $L[\emptyset_i(t,q)] = \frac{\partial \emptyset_i(t,q)}{\partial t}$. From (11) the nonlinear operators are

$$N_1[\emptyset_1(t,q)] = \frac{\partial \emptyset_1(t,q)}{\partial t} - (1 + t + t^2) + \emptyset_2(t,q) + \int_0^t \big(\emptyset_1(s,q) + \emptyset_2(s,q)\big)ds$$

$$N_2[\emptyset_2(t,q)] = \frac{\partial \emptyset_2(t,q)}{\partial t} + (1 + t) - \emptyset_1(t,q) + \int_0^t \big(\emptyset_1(s,q) - \emptyset_2(s,q)\big)ds$$

Let $u_{1,0}(t) = \exp(t)$ and $u_{2,0}(t) = -\exp(t)$

According to Eq. (2) and Eq. (7) with

$$R_{1,m-1}(t) = u_{1,m-1}' - \left(1 - \frac{k_m}{n}\right)(1 + t + t^2) + u_{2,m-1} + \int_0^t (u_{1,m-1} + u_{2,m-1})ds$$

$$R_{2,m-1}(t) = u_{2,m-1}' - \left(1 - \frac{k_m}{n}\right)(1 + t) - u_{1,m-1} + \int_0^t (u_{1,m-1} - u_{2,m-1})ds$$

The solution of Eq. (7) for $m \geq 1$ becomes

$$u_{i,m}(t) = k_m u_{i,m-1}(t) + h \int R_{i,m-1}(t)\, dt + b_i \ , i = 1,2$$

Hence, we obtain

$$u_{1,1} = h(-t - \frac{t^2}{2} - \frac{t^3}{3})$$

$$u_{2,1} = h(-t + \frac{t^2}{2})$$

$$u_{1,2} = hn(-t - \frac{t^2}{2} - \frac{t^3}{3}) + h(-ht - ht^2 - \frac{ht^3}{2} - \frac{ht^5}{60})$$

$$u_{2,2} = hn(-t + \frac{t^2}{2}) + h(-ht + ht^2 + \frac{ht^3}{6} - \frac{ht^5}{60})$$

$$\vdots$$

Then the series solution of q- HAM is

$$u_i(t,n,h) \cong U_{i,M}(t,n,h) = \sum_{k=0}^{M} u_{i,k}(t,n,h)\left(\frac{1}{n}\right)^k \qquad (12)$$

The 4th order Liao's optimal HAM (OHAM) approximation solutions are [29]

$$U_{1,4(\text{OHAM})} = e^t + 0.999994t + 0.000227256t^2 + 0.00234468t^3 - 0.00705523t^4 - 0.0098502t^5 + 0.000470348t^6 + 0.000782842t^7 - 0.00000900634t^9$$

$$U_{2,4(\text{OHAM})} = -e^t + 0.999994t + 0.000227256t^2 + 0.002193178t^3 + 0.0104494t^4 - 0.00702811t^5 - 0.000470348t^6 + 0.000648456t^7 - 0.00000900634t^9$$

It is found that

$$\triangle_{1,1} = \frac{37}{10} + \frac{336059h^2}{45360n^2} + \frac{3287h}{315n}\ ,$$

$$\triangle_{2,1} = \frac{1}{3} + \frac{4687h^2}{9072n^2} + \frac{23h}{90n}\ ,$$

$$\triangle_{1,2} = \frac{37}{10} + \frac{346114261h^4}{32432400n^4} + \frac{44355323h^3}{1247400n^3} + \frac{95561h^2}{2268n^2} + \frac{6574h}{315n}$$

$$\triangle_{2,2}=\frac{1}{3}+\frac{811787h^4}{356400n^4}+\frac{148529h^3}{37800n^3}+\frac{20969h^2}{11340n^2}+\frac{23h}{45n}$$

$\Delta_{i,m}(t)$, $(i=1,2),(m=3,4,\ldots)$ can be calculated similarly.

Figures 1-5 show the h-curves of the square residual error $\Delta_{i,m}(i=1,2)$ given by 4th order approximation to determine the valid region of h. The values of $h$that corresponds to the minimum of the square residual errors is an optimal values of h.

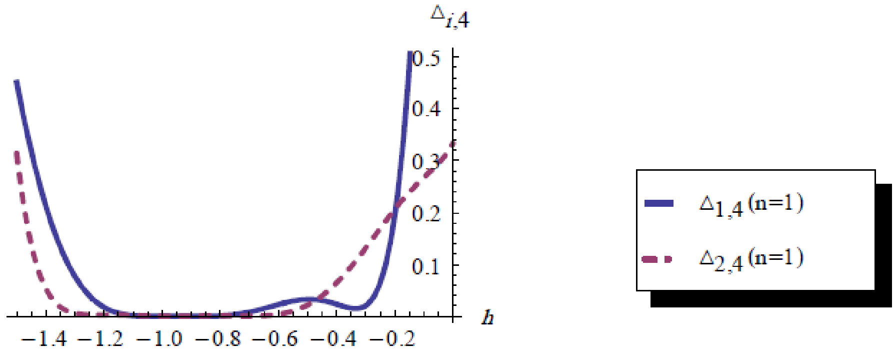

Figure 1. The residual errors $\Delta_{i,4}$ $(i=1,2)$ for the HAM (q-HAM; $n=1$) of system (11).

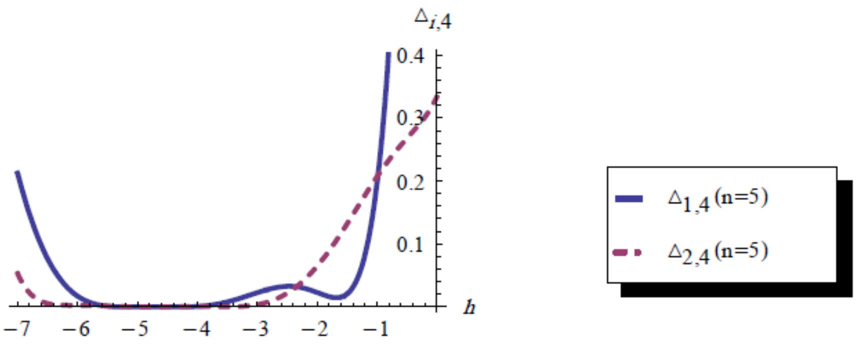

Figure 2. The residual errors $\Delta_{i,4}$ $(i=1,2)$ for the (q-HAM; $n=5$) of system (11).

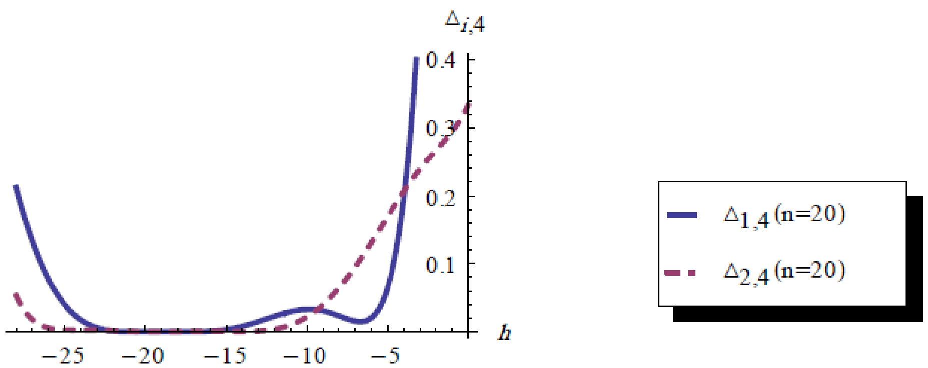

Figure 3. The residual errors $\Delta_{i,4}$ $(i=1,2)$ for the (q-HAM; $n=20$) of system (11).

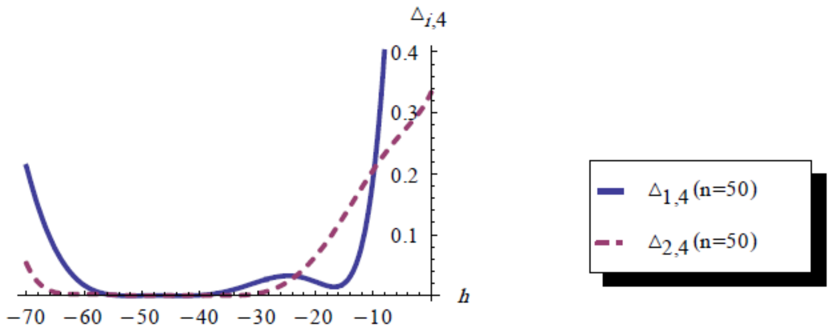

Figure 4. The residual errors $\Delta_{i,4}$ $(i = 1, 2)$ for the (q-HAM; $n = 50$) of system (11).

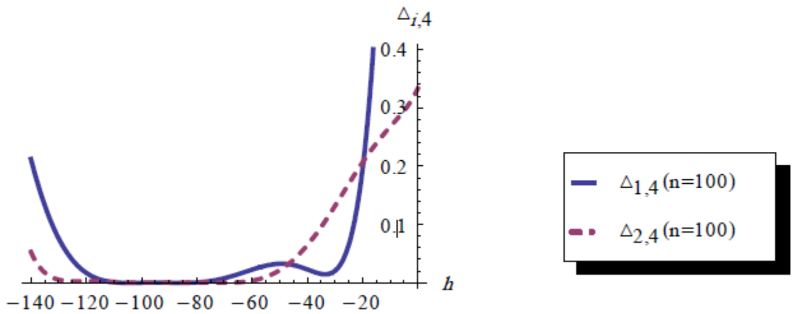

Figure 5. The residual errors $\Delta_{i,4}$ $(i = 1, 2)$ for the (q-HAM; $n = 100$) of system (11).

Tables 1 and 2 show the comparison of $U_{1,4}$, $U_{2,4}$ given by Liao's optimal HAM (OHAM), HAM (q-HAM;$n = 1$) and q-HAM at different values of $n > 1$ and $U_{1,6}$,$U_{2,6}$ given by q-HAM $(n = 100)$ with the solutions $u_1$and u_2. Figures (6-9) illustrate that the highly decreasing of the absolute errors by taking more terms into consideration.

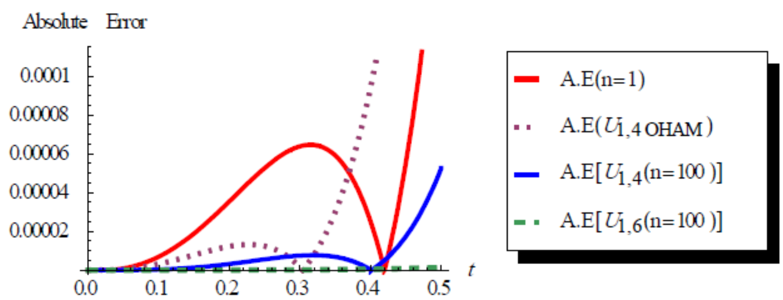

Figure 6. Absolute error of $U_{1,4}$ Liao's optimal HAM (OHAM), $U_{1,4}$HAM (q-HAM; $n = 1$), $U_{1,4}$ q-HAM ($n = 100$) and $U_{1,6}$ q-HAM ($n = 100$) for the system (11) at$0 \le t \le 0.5$ using ($h = -0.9326$, $h = -101.49$) respectively.

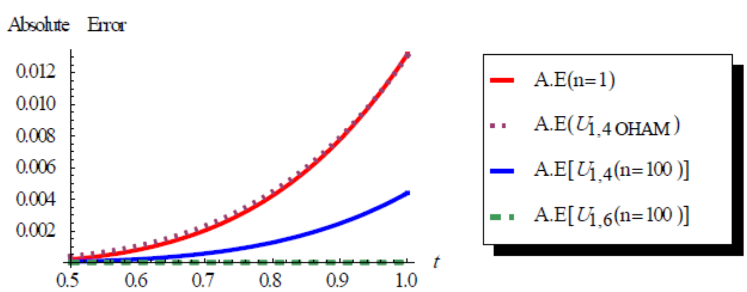

Figure 7. Absolute error of $U_{1,4}$ Liao's optimal HAM (OHAM), $U_{1,4}$ HAM (q-HAM;$n = 1$), $U_{1,4}$ q-HAM ($n = 100$) and $U_{1,6}$q-HAM ($n = 100$) for the system (11) at $0.5 \le t \le 1$ using ($h = -0.9326$, $h = -101.49$) respectively.

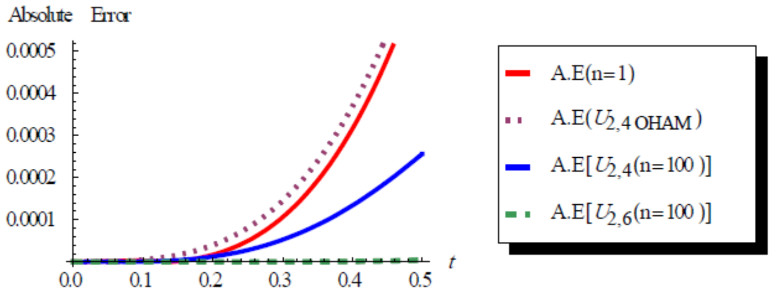

Figure 8. **Absolute error of $U_{2,4}$ Liao's optimal HAM (OHAM) ,$U_{2,4}$ HAM (q-HAM; $n=1$), $U_{2,4}$ q-HAM ($n=100$) and $U_{2,6}$ q-HAM ($n=100$) for the system (11) at $0 \le t \le 0.5$ using ($h=-0.9326$, $h=-96.25$) respectively.**

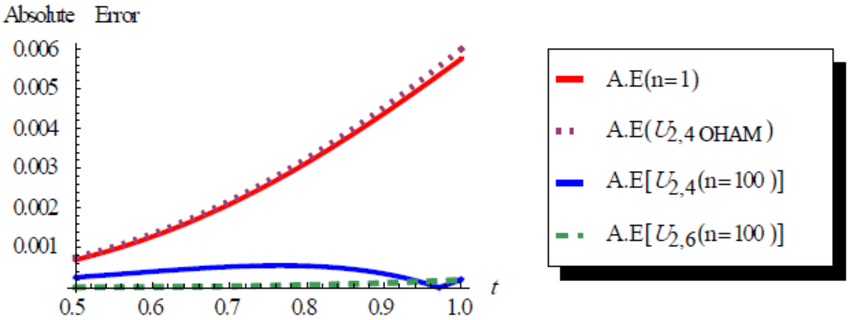

Figure 9. Absolute error of $U_{2,4}$ Liao's optimal HAM (OHAM), $U_{2,4}$ HAM (q-HAM;$n=1$), $U_{2,4}$ q-HAM ($n=100$) and $U_{2,6}$ q-HAM ($n=100$) for the system (11) at $0 \le t \le 0.5$ using ($h=-0.9326$, $h=-96.25$) respectively.

Table 1. Comparison of $U_{1,4}$ given by OHAM, HAM (q-HAM ; $n=1$), q-HAM ($n=5,20,50,100$)and $U_{1,6}$ of q-HAM ($n=100$) with the solution u_1

t	$U_{1,4}$ OHAM	$U_{1,4}$ ($n=1;h=-0.9326$)	$U_{1,4}$ ($n=5;h=-5.001$)	$U_{1,4}$ ($n=20;h=-20.1$)	$U_{1,4}$ ($n=50;h=-50.5$)	$U_{1,4}$ ($n=100;h=-101.49$)	u_1	$U_{1,6}$ ($n=100;h=-101.49$)
0	1	1	1	1	1	1	1	1
0.2	1.421412	1.42144	1.4214	1.4214	1.4214	1.4214	1.4214	1.4214
0.4	1.89173	1.89185	1.89174	1.89177	1.89179	1.89182	1.89182	1.89182
0.6	2.42107	2.42131	2.4215	2.42162	2.42176	2.42191	2.42212	2.42212
0.8	3.02105	3.02131	3.02299	3.02337	3.02381	3.02427	3.02554	3.02554
1	3.70519	3.70508	3.71077	3.71171	3.71277	3.71388	3.718281	3.7183

Table 2. Comparison of $U_{2,4}$ given by OHAM, HAM (q-HAM ; $n=1$), q-HAM ($n=5,20,50,100$) and $U_{2,6}$ of by q-HAM ($n=100$) with the solution u_2

t	$U_{2,4}$ OHAM	$U_{2,4}$ ($n=1;h=-0.9326$)	$U_{2,4}$ ($n=5;h=-4.7$)	$U_{2,4}$ ($n=20;h=-19.01$)	$U_{2,4}$ ($n=50;h=-48.001$)	$U_{2,4}$ ($n=100;h=-96.25$)	u_2	$U_{2,6}$ ($n=100;h=-96.25$)
0	-1	-1	-1	-1	-1	-1	-1	-1
0.2	-1.02136	-1.02139	-1.02139	-1.02139	-1.02139	-1.02139	-1.0214	-1.0214
0.4	-1.09146	-1.09151	-1.09155	-1.09161	-1.09167	-1.09169	-1.09182	-1.09183
0.6	-1.22076	-1.22084	-1.22105	-1.22136	-1.22164	-1.22172	-1.22212	-1.22213
0.8	-1.42229	-1.4224	-1.42303	-1.42395	-1.42478	-1.425	-1.42554	-1.4256
1	-1.71228	-1.71253	-1.71398	-1.71608	-1.71799	-1.71849	-1.718281	-1.71847

Example 3.2. Consider the following problem of nonlinear integro-differential equation [29]

$$u^{'}(t) = -1 + \int_0^t u^2(s)ds\, , u(0) = 0, t \in [0,1] \tag{13}$$

Assume that $L[\emptyset(t,q)] = \frac{\partial\emptyset(t,q)}{\partial t}$. From (13) the nonlinear operator is

$$[\emptyset(t,q)] = \frac{\partial\emptyset(t,q)}{\partial t} + 1 - \int_0^t \emptyset^2(s,q)ds.$$

Let $u_0(t) = -t$.

According to Eq. (2) and Eq. (7) with $R_{m-1}(t) = u_{m-1}^{'} + \left(1 - \frac{k_m}{n}\right) - \int_0^t \sum_{i=0}^{m-1} u_i(s)u_{m-1-i}(s)\, ds$ the solution Eq. (7) for $m \geq 1$ becomes

$$u_m(t) = k_m u_{m-1}(t) + h\int R_{m-1}(t)\, dt + b,$$

Hence, we obtain

$$u_1 = -\frac{ht^4}{12},$$
$$u_2 = -\frac{1}{12}hnt^4 + h(-\frac{ht^4}{12} - \frac{ht^7}{252}),$$
$$\vdots$$

Then the series solution of q- HAM is

$$u(t,n,h) \cong U_M(t,n,h) = \sum_{k=0}^{M} u_k(t,n,h)\left(\frac{1}{n}\right)^k \tag{14}$$

It is found that

$$\triangle_1 = \frac{1}{63} + \frac{h^4}{31912704n^4} + \frac{205h^3}{4852224n^3} + \frac{31531h^2}{1769040n^2} + \frac{127h}{3780n}$$

$$\triangle_2 = \frac{1}{63} + \frac{297235061h^8}{8398362595392000n^8} + \frac{173605561h^7}{632134819008000n^7} + \frac{9482512903h^6}{195660777312000n^6} + \frac{102775727h^5}{372687194880n^5} + \frac{8547857h^4}{418279680n^4} + \frac{1070983h^3}{14152320n^3} + \frac{173h^2}{1620n^2} + \frac{127h}{1890n}$$

$\Delta_m(t)$, $(m = 3,4,\dots)$ can be calculated similarly.

Figures 10-14 show the h-curves of the square residual error Δ_m given by (1^{th}, 2^{th}, 3^{th}, 4^{th}) order approximation to determine the valid region of h. The values of h that corresponds to the minimum of the square residual errors is an optimal values of h. Table 3 shows the comparison of OHAM), HAM and q-HAM at different values of $n > 1$. Clearly, the q-HAM $n > 1$ has more accuracy than HAM and OHAM.

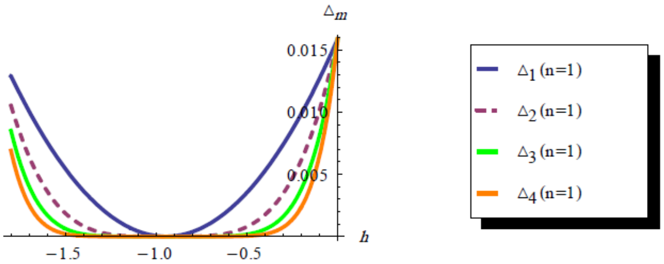

Figure 10. The residual errors Δ_m $(m = 1,2,3,4)$ of HAM.

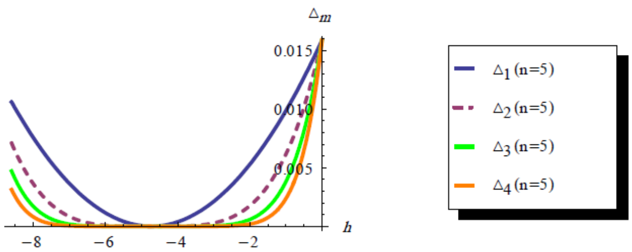

Figure 11. The residual errors Δ_m $(m = 1,2,3,4)$ of q-HAM $(n = 5)$.

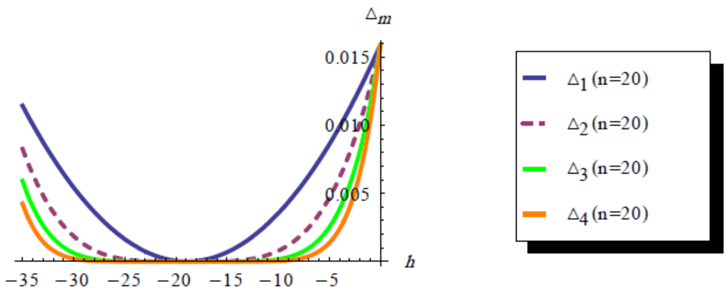

Figure 12. The residual errors Δ_m $(m = 1, 2, 3, 4)$ of q-HAM $(n = 20)$.

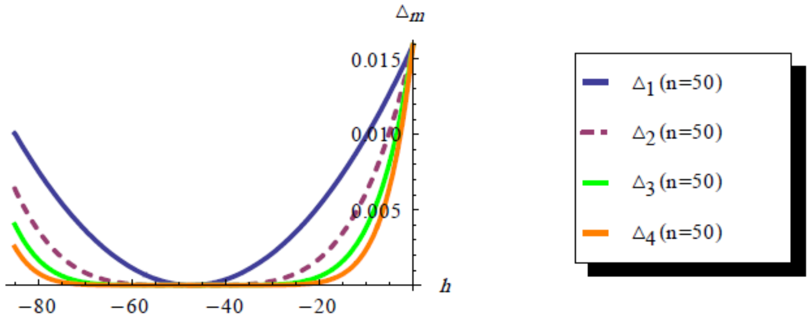

Figure 13. The residual errors Δ_m $(m = 1, 2, 3, 4)$ of q-HAM $(n = 50)$.

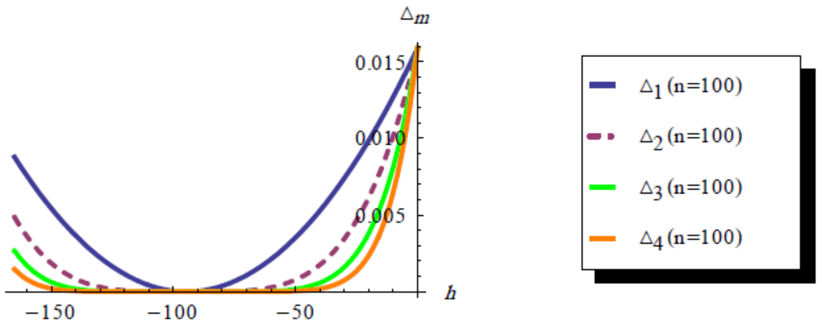

Figure 14. The residual errors Δ_m $(m = 1, 2, 3, 4)$ of q-HAM $(n = 100)$.

Table 3. Comparison between residuals of OHAM, HAM and q-HAM $(n > 1)$

m	$E_m(OHAM)$	$\Delta_m(n = 1; h_m)$	$\Delta_m(n = 5; h_m)$	$\Delta_m(n = 20; h_m)$	$\Delta_m(n = 50; h_m)$	$\Delta_m(n = 100; h_m)$
1	0.588916e-5	0.470089e-5; -0.942067	0.466546e-5; -4.745	0.455403e-5; -18.87	0.453548e-5; -47.38	0.447006e-5; -94.567
2	0.182630e-8	0.161694e-8; -0.965	0.1553741e-8; -4.849	0.14836e-8; -19.39	0.132012e-8; -48.42	0.128842e-8; -96.75
3	0.247549e-12	0.229395e-12; -0.9717	0.211484e-12; -4.86	0.199042e-12; -19.445	0.185862e-12; -48.69	0.17559e-12; -97.32
4	0.671736e-16	0.832667e-16; -0.9746	0.659195e-16; -4.8862	0.555112e-16; -19.5257	0.277556e-16; -48.882	0.346945e-17; -97.6296

4. Conclusion

In this work, we introduced the comparative study of q-HAM with the Liao's optimal HAM (OHAM). We solved two examples to illustrate the differences between these methods. The outcomes show that the q-HAM was better than the OHAM.

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