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A Note on Two (3+1)-Dimensional Gardner-Type Equation with Multiple Kink Solutions

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Abstract

It is difficult or impossible to find the exact solutions for many partial differential equations. In recent years, a variety of efficient and practical methods have been proposed by mathematicians. This article investigates the exact solutions of partial differential equations. The Gardner equation is well known in literature and is applicable in various branches of physics. The Gardner equation belongs to the category of non-linear partial differential equations. This equation and its generalizations are used in many areas of applications, such as hydrodynamics, plasma physics, and quantum field theory. The Gardner-type equations are the useful model to understand the propagation of negative ion acoustic plasma waves. These type of equations can be derived from the system of plasma motion equations in one dimension with arbitrarily charged cold ions and inertia neglected isothermal electrons. Numerous numerical and analytical methods have been used to study this equation. That proves the importance of this equation. In this paper, we examine the exact travelling wave solutions of the Gardner equation. The Hirota's bilinear method and the Cole-Hopf transformation is used to obtain an elegant formula for the exact travelling wave solution. We demonstrate a correct formula for exact solutions. Mathematica software and the standart LaTex tools was used to perform the computations. The suggested approach can be used in other real-world models in science and engineering.

Keywords

Gardner-type equation, Hirota bilinear method, soliton solutions, kink solutions

1. Introduction

In the present paper by "Gardner equation", we assume an equation in integro-differential form

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y = 0, \quad (1)$$

where $\sigma^2 = \pm 1$, α, β are arbitrary constants and ∂_x^{-1} is the inverse of ∂_x with $\partial_x^{-1}\partial_x = \partial_x \partial_x^{-1} = 1$. The equation (1) is widely used in plasma physics, fluid physics, quantum field theory and different branches of physics [1-7]. There are different methods, such as the inverse scattering method, the Casorati and Grammian determinant solutions, and the Hirota's direct method for solving equation (1) see, for example, [1,10]. For the travelling wave solutions of the Gardner-type equation in terms of corresponding ordinary differential equations see [12,13].

The more general forms of the equation (1) were examined in [11] by using the simplified Hirota's method:

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y + 3\sigma^2 \partial_x^{-1} u_{zz} = 0, \quad (2)$$

$$u_t+6\beta u u_x + u_{xxx} - \frac{3\alpha^2 u^2 u_x}{2} + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y + 3\sigma^2 \partial_x^{-1} u_{zz} - 3\alpha\sigma u_x \partial_x^{-1} u_z = 0. \quad (3)$$

In this work, we analyze the equation (2) and find an elegant formula for the exact travelling wave solution.

2. The travelling-wave solutions of the generalized Gardner-type equation

In this Section, we first derive kink solutions for the Gardner-type equation (2), where

$$(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t)dt, \quad (4)$$

with decaying condition at infinity.

The transformation

$$u(x,y,z,t) = v_x(x,y,z,t) \quad (5)$$

translates (2) into the equation

$$v_{xt} + 6\beta v_x v_{xx} + v_{xxxx} - \frac{3}{2}\alpha^2 v_x^2 v_{xx} + 3\sigma^2 v_{yy} - 3\alpha\sigma v_{xx} v_y + 3\sigma^2 v_{zz} = 0. \quad (6)$$

To obtain the exact solution we substitute

$$v(x,y,z,t) = e^{kx+ry+sz-ct} \quad (7)$$

into the linear terms of (6), which gives c as

$$c = \frac{1}{k}(3r^2\sigma^2 + 3s^2\sigma^2 + k^4), \quad (8)$$

and the dispersion variable as

$$\theta = kx + ry + sz - \frac{1}{k}(3r^2\sigma^2 + 3s^2\sigma^2 + k^4)t. \quad (9)$$

Next we use the Cole-Hopf transformation

$$u(x,y,z,t) = R(\ln f(x,y,z,t))_x \quad (10)$$

or

$$v(x,y,z,t) = R(\ln f(x,y,z,t)). \quad (11)$$

To get the auxiliary function $f(x,y,z,t)$, we use the simplified Hirota's method: Substituting

$$f(x,y,z,t) = 1 + e^{kx+ry+sz-\frac{1}{k}(3r^2\sigma^2+3s^2\sigma^2+k^4)t} \quad (12)$$

into (6), [11] gives the single-kink solution:

$$u(x,y,z,t) = v_x(x,y,z,t))$$
$$= \frac{2ke^{kx+\frac{k(2\beta-\alpha k)}{\sigma\alpha}y+sz-\left(k^3+\frac{3k(2\beta-\alpha k)^2}{\alpha^2}+3\alpha^2\frac{s^2}{k}\right)t}}{\alpha\left(1+e^{kx+\frac{k(2\beta-\alpha k)}{\sigma\alpha}y+sz-\left(k^3+\frac{3k(2\beta-\alpha k)^2}{\alpha^2}+3\alpha^2\frac{s^2}{k}\right)t}\right)}, \quad (13)$$

(Eq. (20) in [11]). The author [11] then used the same approach to establish the similar formula for the double-kink solution

$$u = \frac{2\sum_{i=1}^{2} k_i e^{k_i x+\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha}y+s_i z-\left(k_i{}^3+\frac{3k_i(2\beta-\alpha k_i)^2}{\alpha^2}+3\alpha^2\frac{s_i^2}{k_i}\right)t}}{\alpha\left(1+\sum_{i=1}^{2} k_i e^{k_i x+\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha}y+s_i z-\left(k_i{}^3+\frac{3k_i(2\beta-\alpha k_i)^2}{\alpha^2}+3\alpha^2\frac{s_i^2}{k_i}\right)t}\right)} \quad (14)$$

Let us show that the formula (14) is not correct and prove the correct formula.

Example 2.1. Let $k_1=1$, $k_2=2$, $s_1=1$, $s_2=0$, $\alpha=2$, $\beta=\sigma=1$, it is easy to show that (14) is not the solution of the equation (6). Indeed

$$u=\frac{2\left(e^{x+\frac{(2-2)}{2}y+z-(1+\frac{3(2-2)^2)}{4}+3\frac{1}{1})t}+2e^{2x+\frac{2(2-4)}{2}y-(8+\frac{6(2-4)^2)}{4})t}\right)}{2\left(1+e^{x+\frac{(2-2)}{2}y+z-(1+\frac{3(2-2)^2)}{4}+3\frac{1}{1})t}+e^{2x+\frac{2(2-4)}{2}y-(8+\frac{6(2-4)^2)}{4})t}\right)}$$

$$=\frac{2e^{2x-14t-2y}+e^{x-5t+z}}{e^{2x-14t-2y}+e^{x-5t+z}+1}=v_x,$$

$$v=ln(1+e^{2x-14t-2y}+e^{x-5t+z}),$$

and simplifying the expression

$$v_{xt}+6\beta v_x v_{xx}+v_{xxxx}-\frac{3}{2}\alpha^2 v_x^2 v_{xx}+3\sigma^2 v_{yy}-3\alpha\sigma v_{xx}v_y+3\sigma^2 v_{zz}$$

we obtain

$$\frac{6(e^{3x-18t-2y+z}+2e^{5x-32t-4y+z}+e^{7x-46t-6y+z}+e^{4x-22t-2y+2z}+e^{5x-26t-2y+3z}+2e^{6x-36t-4y+2z})}{(e^{2x-14t-2y}+e^{x-4t+z}+1)^4}.$$

We used standard LaTex techniques to simplify the expression in the left hand side of (6). This expression is positive for any choice of x, y, z and t. That is (14) is not the solution of the equation (6).

Now to find a correct double-kink solution we set

$$v(x,y,z,t)=R(\ln(1+e^{k_1x+r_1y+s_1z+c_1t}+e^{k_2x+r_2y+s_2z+c_2t}), \tag{15}$$

$$v_x=R\frac{k_1e^{xk_1+\cdots}+k_2e^{xk_2+\cdots}}{1+e^{xk_1+\cdots}+e^{xk_2+\cdots}},$$

$$v_{xt}=R\frac{c_1k_1e^{xk_1+\cdots}+c_2k_2e^{xk_2+\cdots}+\sum_{i,j=1,2}c_ik_je^{xk_1+\cdots}e^{xk_2+\cdots}}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2},$$

$$6\beta v_x v_{xx}=6\beta R^2\frac{k_1e^{xk_1+\cdots}+k_2e^{xk_2+\cdots}}{1+e^{xk_1+\cdots}+e^{xk_2+\cdots}}$$

$$\frac{{k_1}^2e^{xk_1+\cdots}+{k_2}^2e^{xk_2+\cdots}+{k_1}^2e^{xk_1+\cdots}e^{xk_2+\cdots}+{k_2}^2e^{xk_1+\cdots}e^{xk_2+\cdots}-2k_1k_2e^{xk_1+\cdots}e^{xk_2+\cdots}}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2},$$

$$\begin{aligned}v_{xxxx}=\frac{R}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^4}&({k_1}^4e^{xk_1+\cdots}-4{k_1}^4e^{2xk_1+\cdots}+{k_2}^4e^{xk_2+\cdots}\\&+{k_1}^4e^{3xk_1+\cdots}-4{k_2}^4e^{2xk_2+\cdots}+{k_2}^4e^{3xk_2+\cdots}+3{k_1}^4e^{xk_1+\cdots}e^{xk_2+\cdots}+3{k_2}^4e^{xk_1+\cdots}e^{xk_2+\cdots}\\&+3k_1^4e^{xk_1+\cdots}e^{2xk_2+\cdots}-8k_1^4e^{xk_2+\cdots}e^{2xk_1+\cdots}+k_1^4e^{xk_1+\cdots}e^{3xk_2+\cdots}+k_1^4e^{xk_2+\cdots}e^{3xk_1+\cdots}\\&-8k_2^4e^{xk_1+\cdots}e^{2xk_2+\cdots}+3k_2^4e^{xk_2+\cdots}e^{2xk_1+\cdots}-4k_1^4e^{2xk_1+\cdots}e^{2xk_2+\cdots}+k_2^4e^{3xk_2+\cdots}e^{xk_1+\cdots}\\&+k_2^4e^{3xk_1+\cdots}e^{xk_2+\cdots}-4k_2^4e^{2xk_2+\cdots}e^{2xk_1+\cdots}-4k_1k_2^3e^{xk_1+\cdots}e^{xk_2+\cdots}\\&-4k_1^3k_2e^{xk_1+\cdots}e^{xk_2+\cdots}+16k_2^3k_1e^{2xk_2+\cdots}e^{xk_1+\cdots}-8k_1k_2^3e^{2xk_1+\cdots}e^{xk_2+\cdots}\\&-8k_1^3k_2e^{xk_1+\cdots}e^{2xk_2+\cdots}+16k_1^3k_2e^{xk_2+\cdots}e^{2xk_1+\cdots}-4k_1k_2^3e^{xk_1+\cdots}e^{3xk_2+\cdots}\\&-4k_2^3k_1e^{3xk_1+\cdots}e^{xk_2+\cdots}-4k_1^3k_2e^{3xk_2+\cdots}e^{xk_1+\cdots}-4k_2k_1^3e^{3xk_1+\cdots}e^{xk_2+\cdots}\\&+16k_2^3k_1e^{2xk_1+\cdots}e^{2xk_2+\cdots}+16k_1^3k_2e^{2xk_2+\cdots}e^{2xk_1+\cdots}-6k_1^2k_2^2e^{xk_1+\cdots}e^{xk_2+\cdots}\\&+6k_1^2k_2^2e^{xk_1+\cdots}e^{3xk_2+\cdots}+6k_1^2k_2^2e^{xk_2+\cdots}e^{3xk_1+\cdots}-24k_1^2k_2^2e^{2xk_1+\cdots}e^{2xk_2+\cdots}),\end{aligned}$$

$$-\frac{3}{2}\alpha^2v_x^2v_{xx}=-\frac{2}{3}\alpha^2R^3\frac{(k_1e^{xk_1+\cdots}+k_2e^{xk_2+\cdots})^2}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2}$$

$$\frac{k_1^2e^{xk_1+\cdots}+k_2^2e^{xk_2+\cdots}+k_1^2e^{xk_1+\cdots}e^{xk_2+\cdots}+k_2^2e^{xk_1+\cdots}e^{xk_2+\cdots}-2k_1k_2e^{xk_1+\cdots}e^{xk_2+\cdots}}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2},$$

$$3\sigma^2v_{yy}=3\sigma^2R\frac{r_1^2e^{xk_1+\cdots}+r_2^2e^{xk_2+\cdots}+r_1^2e^{xk_1+\cdots}e^{xk_2+\cdots}+r_2^2e^{xk_1+\cdots}e^{xk_2+\cdots}-2r_1r_2e^{xk_1+\cdots}e^{xk_2+\cdots}}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2},$$

$$-3\alpha\sigma v_{xx} v_y = -3\alpha\sigma R^2 \frac{k_1^2 e^{xk_1+\cdots} + k_2^2 e^{xk_2+\cdots} + k_1^2 e^{xk_1+\cdots} e^{xk_2+\cdots} + k_2^2 e^{xk_1+\cdots} e^{xk_2+\cdots} - 2k_1k_2 e^{xk_1+\cdots} e^{xk_2+\cdots}}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2}$$

$$\frac{r_1 e^{xk_1+\cdots} + r_2 e^{xk_2+\cdots}}{1+e^{xk_1+\cdots}+e^{xk_2+\cdots}},$$

$$3\sigma^2 v_{zz} = 3\sigma^2 R \frac{s_1^2 e^{xk_1+\cdots} + s_2^2 e^{xk_2+\cdots} + ks_1^2 e^{xk_1+\cdots} e^{xk_2+\cdots} + s_2^2 e^{xk_1+\cdots} e^{xk_2+\cdots} - 2s_1s_2 e^{xk_1+\cdots} e^{xk_2+\cdots}}{(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^2}.$$

Substituting into (6), multiplying by $(1+e^{xk_1+\cdots}+e^{xk_2+\cdots})^4$ and equalizing the coefficients of $e^{mxk_1+\cdots}$ and $e^{nxk_2+\cdots}$, m=1, 2, 3, we obtain the equations

$$\begin{gathered} k_1^4 + 3\sigma^2 r_1^2 + 3\sigma^2 s_1^2 + c_1 k_1 = 0, \\ k_2^4 + 3\sigma^2 r_2^2 + 3\sigma^2 s_2^2 + c_2 k_2 = 0, \end{gathered} \tag{16}$$

$$\begin{gathered} 2k_1^2 - 2R\beta k_1 + R\alpha\sigma r_1 = 0, \\ 2k_2^2 - 2R\beta k_2 + R\alpha\sigma r_2 = 0' \end{gathered} \tag{17}$$

$$\begin{gathered} -4\beta k_1 + 2\alpha\sigma r_1 + R\alpha^2 k_1^2 = 0, \\ -4\beta k_2 + 2\alpha\sigma r_2 + R\alpha^2 k_2^2 = 0, \end{gathered} \tag{18}$$

which gives

$$R = \frac{2}{\alpha}, \qquad r_1 = \frac{1}{\alpha\sigma}(2\beta k_1 - \alpha k_1^2), \qquad r_2 = \frac{1}{\alpha\sigma}(2\beta k_2 - \alpha k_2^2).$$

Coefficients of mixed products $e^{mxk_1+\cdots}$ $e^{nxk_2+\cdots}$ can be translated into the equation

$$(k_1-k_2)^4 + 3\sigma^2(r_1-r_2)^2 + 3\sigma^2(s_1-s_2)^2 + (c_1-c_2)(k_1-k_2) = 0, \tag{19}$$

and we obtain the next system for unknowns s_2 and c_2 (for given k_1, s_1, $r_1 = \frac{1}{\alpha\sigma}(2\beta k_1 - \alpha k_1^2)$, $r_2 = \frac{1}{\alpha\sigma}(2\beta k_2 - \alpha k_2^2)$, $c_1 = -k_1^3 - 3\frac{\sigma^2}{k_1} r_1^2 - 3\frac{\sigma^2}{k_1} s_1^2$, k_2)

$$\begin{gathered} k_1^4 + 3\sigma^2(r_1^2 + s_1^2) + c_1 k_1 = 0, \\ k_2^4 + 3\sigma^2(r_2^2 + s_2^2) + c_2 k_2 = 0, \\ (k_1-k_2)^4 + 3\sigma^2(r_1-r_2)^2 + 3\sigma^2(s_1-s_2)^2 + (c_1-c_2)(k_1-k_2) = 0 \end{gathered} \tag{20}$$

or

$$\begin{gathered} c_1 = -k_1^3 - 3\frac{\sigma^2}{k_1} r_1^2 - 3\frac{\sigma^2}{k_1} s_1^2 \\ c_2 = -k_2^3 - 3\frac{\sigma^2}{k_2} r_2^2 - 3\frac{\sigma^2}{k_2} s_2^2 \\ c_1 - c_2 = -(k_1-k_2)^3 - 3\sigma^2 \frac{(r_1-r_2)^2 + (s_1-s_2)^4}{k_1-k_2}. \end{gathered} \tag{21}$$

First and second equations replaced into the third one gives

$$\begin{gathered} -k_1^3 - 3\frac{\sigma^2}{k_1} r_1^2 - 3\frac{\sigma^2}{k_1} s_1^2 - \left(-k_2^3 - 3\frac{\sigma^2}{k_2} r_2^2 - 3\frac{\sigma^2}{k_2} s_2^2\right) \\ = -(k_1-k_2)^3 - 3\sigma^2 \frac{(r_1-r_2)^2 + (s_1-s_2)^4}{k_1-k_2} \end{gathered} \tag{22}$$

and

$$(\sigma^2(k_1r_2 - k_2r_1)^2 + \sigma^2(k_1s_2 - k_2s_1)^2 - k_1^2k_2^2(k_1-k_2)^2) = 0. \tag{23}$$

Now substituting

$$r_i = \frac{1}{\alpha\sigma}(2\beta k_i - \alpha k_i^2), \qquad i = 1, 2$$

we have

$$\begin{gathered}\sigma^2\left(k_1\frac{1}{\alpha\sigma}(2\beta k_2-\alpha k_2^2)-k_2\frac{1}{\alpha\sigma}(2\beta k_1-\alpha k_1^2)\right)^2\\+\sigma^2(k_1s_2-k_2s_1)^2-k_1^2k_2^2(k_1-k_2)^2\\=\sigma^2(k_1s_2-k_2s_1)^2\end{gathered}\tag{24}$$

and

$$s_2=\frac{k_2}{k_1}s_1.\tag{25}$$

This relationship allows us to find the elegant double-kink solutions. First, we find c_2 from the second equation in (21) and (25):

$$c_2=-k_2^3-3\frac{\sigma^2}{k_2}r_2^2-3\frac{\sigma^2k_2}{k_1}s_1^2$$

and therefore, the exact solution is

$$\begin{aligned}v&=\frac{2}{\alpha}\ln(1+e^{k_1x+\frac{2\beta k_1-\alpha k_1^2}{\alpha\sigma}y+s_1z-(k_1^3+3\frac{\left(2\beta k_1-\alpha k_1^2\right)^2}{\alpha^2k_1}+3\frac{\sigma^2s_1^2}{k_1})t}\\&\quad+e^{k_2x+\frac{2\beta k_2-\alpha k_2^2}{\alpha\sigma}y+\frac{k_2}{k_1}s_1z-(k_2^3+3\frac{\left(2\beta k_2-\alpha k_2^2\right)^2}{\alpha^2k_2}+3\frac{\sigma^2k_2s_1^2}{k_1^2})t}\\&=\frac{2}{\alpha}\ln\left(1+\sum_{i=1}^2e^{k_ix+\frac{2\beta k_i-\alpha k_i^2}{\alpha\sigma}y+\frac{k_i}{k_1}s_1z-(k_i^3+3\frac{\left(2\beta k_i-\alpha k_i^2\right)^2}{\alpha^2k_1}+3\frac{\sigma^2k_is_1^2}{k_1^2})t}\right)\end{aligned}\tag{26}$$

Now to find a wave travelling solution of the equation (2), we take $k_1=n_1+in_2$, $k_2=n_1-in_2$, $s_1=s+\frac{n_2}{n_1}si$, $n_1,n_2,s\in R$ and consider the function

$$\frac{2}{\alpha}\ln\left(1+\sum_{i=1}^{2}e^{k_ix+\frac{2\beta k_i-\alpha k_i^2}{\alpha\sigma}y+\frac{k_i}{k_1}s_1z-(k_i^3+3\frac{\left(2\beta k_i-\alpha k_i^2\right)^2}{\alpha^2k_1}+3\frac{\sigma^2k_is_1^2}{k_1^2})t}\right).$$

We have for i=1

$$\begin{gathered}k_1x+\frac{2\beta k_1-\alpha k_1^2}{\alpha\sigma}y+s_1z-(k_1^3+3\frac{(2\beta k_1-\alpha k_1^2)^2}{\alpha^2k_1}+3\frac{\sigma^2s_1^2}{k_1})t\\(n_1+in_2)x+\frac{2\beta(n_1+in_2)-\alpha(n_1+in_2)^2}{\alpha\sigma}y+s(1+\frac{n_2}{n_1}i)z\\-\left((n_1+in_2)^3+3\frac{(2\beta(n_1+in_2)-\alpha(n_1+in_2)^2)^2}{\alpha^2(n_1+in_2)}+\frac{3\sigma^2s^2\left(1+\frac{n_2}{n_1}i\right)^2}{(n_1+in_2)}\right)t\\=n_1x+in_2x+\frac{2\beta n_1+\alpha n_2^2-\alpha n_1^2}{\alpha\sigma}y+\frac{n_2(2\beta-2\alpha n_1)}{\alpha\sigma}yi+s(1+\frac{n_2}{n_1}i)z\\=-\frac{1}{\alpha^2n_1}(3s^2\alpha^2\sigma^2+4\alpha^2n_1^4-12\alpha^2n_1^2n_2^2-12\alpha\beta n_1^3+12\alpha\beta n_1n_2^2+12\beta^2n_1^2)t\\-\frac{1}{\alpha^2n_1^2}n^2(3s^2\alpha^2\sigma^2+12\alpha^2n_1^4-4\alpha^2n_1^2n_2^2-24\alpha\beta n_1^3+12\beta^2n_1^2)ti,\end{gathered}$$

and for i=2

$$\begin{gathered}k_2x+\frac{2\beta k_2-\alpha k_2^2}{\alpha\sigma}y+\frac{k_2s_1}{k_1}z-(k_2^3+3\frac{(2\beta k_2-\alpha k_2^2)^2}{\alpha^2k_2}+3\frac{k_2\sigma^2}{k_1^2}s_1^2)t\\=(n_1-in_2)x+\frac{2\beta(n_1-in_2)-\alpha(n_1-in_2)^2}{\alpha\sigma}y+\frac{(n_1-in_2)s(1+\frac{n_2}{n_1}i)}{n_1+in_2}z\\-((n_1-in_2)^3+\frac{3(2\beta(n_1-in_2)-\alpha(n_1-in_2)^2)^2}{\alpha^2(n_1-in_2)}\end{gathered}$$

$$+\frac{3(n_1 - in_2)\sigma^2 s^2 (1 + (n_2/n_1)i)^2}{(n_1 + in_2)^2} t$$

$$= n_1 x - in_2 x + \frac{2\beta n_1 + \alpha n_2^2 - \alpha n_1^2}{\alpha\sigma} y - \frac{n_2(2\beta - 2\alpha n_1)}{\alpha\sigma} yi$$

$$+s(1 - (n_2/n_1)i)z$$

$$-\frac{1}{\alpha^2 n_1}(3s^2\alpha^2\sigma^2 + 4\alpha^2 n_1^4 - 12\alpha^2 {n_1}^2 n_2^2 - 12\alpha\beta n_1^3 + 12\alpha\beta n_1 n_2^2 + 12\beta^2 n_1^2)t$$

$$+\frac{n^2}{\alpha^2 n_1^2}(3s^2\alpha^2\sigma^2 + 12\alpha^2 n_1^4 - 4\alpha^2 n_1^2 n_2^2 - 24\alpha\beta n_1^3 + 12\beta^2 n_1^2)ti.$$

This in turn gives

$$\sum_{n=1}^{2} e^{k_i x + \frac{2\beta k_i - \alpha k_i^2}{\alpha\sigma} y + \frac{k_i s_1}{k_1} z - (k_i^3 + 3\frac{\left(2\beta k_i - \alpha k_i^2\right)^2}{k_i \alpha^2} + 3\frac{k_i \sigma^2}{k_1^2} s_1^2)t}$$

$$= 2\cos(n_1 x + \frac{2\beta n_1 + \alpha n_2^2 - \alpha n_1^2}{\alpha\sigma} y + cz$$

$$-\frac{1}{\alpha^2 n_1}(3s^2\alpha^2\sigma^2 + 4\alpha^2 n_1^4 - 12\alpha^2 {n_1}^2 n_2^2 - 12\alpha\beta n_1^3 + 12\alpha\beta n_1 n_2^2 + 12\beta^2 n_1^2)t$$

$$= 2\cos(n_1 x + \frac{2\beta n_1 + \alpha n_2^2 - \alpha n_1^2}{\alpha\sigma} y + cz$$

$$-12t\left(\frac{\beta^2 n_1}{\alpha^2} + \frac{n_1^3}{3} + \frac{\sigma^2 s^2}{4n^1} - \frac{\beta}{\alpha} n_1^2 - n_2^2 n_1 - \frac{\beta}{\alpha} n_2^2\right)),$$

and the travelling-wave solution

$$u(x, y, z, t) = \frac{4n_1 sin(n_1 x + \phi(n_1, n_2)y + sz - \psi(n_1, n_2, s)t)}{\alpha(1 + 2cos(n_1 x + \phi(n_1, n_2)y + sz - \psi(n_1, n_2, s)t)}$$

where

$$\phi(n_1, n_2) \equiv \frac{2\beta n_1 - \alpha n_1^2 + \alpha n_2^2}{\alpha\sigma},$$

$$\psi(n_1, n_2, s) \equiv 12\left(\frac{\beta^2 n_1}{\alpha^2} + \frac{n_1^3}{3} + \frac{\sigma^2 s}{4n_1} - \frac{\beta}{\alpha} n_1^2 - n_2^2 n_1 - \frac{\beta}{\alpha} n_2^2\right).$$

3. Conclusions

This paper has described how the travelling-wave solutions can be received from double-kink solutions. We found the necessary conditions for double kink solution to exist. The dispersion relations were also derived.

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