Singular Travelling Wave Solutions for a Generalized Camassa Holm Equation

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Abstract

In this paper, we concern the singular traveling waves for a generalized Camassa-Holm equation. Camassa and Holm derived an equation to describe unidirectional propagation of shallow water waves over a flat bottom, which is called Camassa-Holm equation, the Camassa-Holm equation is completely integrable and possesses an infinite number of conservation laws, it has been well investigated in the view of mathematical point and many results were obtained. Furthermore, some authors have studied a generalized Camassa-Holm equation. Special solutions play an important role in the research of partial differential equations, and it can be used to describe and explain many phenomena in physics and engineering and so on. A particular kind of product of distributions is introduced and applied to solve non-smooth solutions of this equation. It is proved that, under certain conditions, the distribution solutions such as singular Dirac delta function and Heaviside function exist for the Camassa-Holm equation.

Keywords

Camassa Holm Equation, Product of Distributions, Traveling delta Wave Solution

1. Introduction

Camassa and Holm derived an equation to describe unidirectional propagation of shallow water waves over a flat bottom in [1], which is called Camassa-Holm equation (CH for short),

\[ u_t + 2ku_x - uu_{xx} + 3u_{xx} = 2u_xu_{xx} + uu_{xxx} \]  

(1.1)

Where \( u \) is the fluid velocity in the \( x \) direction (or equivalently the height of the water’s free surface above a flat bottom), \( k \) is a constant related to the critical shallow water wave speed, and it was shown that the CH equation is completely integrable and possesses an infinite number of conservation laws. Actually the equation (1.1) was firstly proposed by Fokas and Fuchssteiner by the method of recursion operators for studying completely integrable generalization of the KdV equation with bi-Hamiltonian structure in [2]. It is quite different from the classic KdV equation that the CH equation has much more phenomena than the former, such as peakons and breaking waves (see [1, 3, 4]). As mentioned in [5], it is interesting to find that, in one mathematical model of shallow water waves, both phenomena of soliton interaction and wave breaking can be discovered. The CH equation has been well investigated in the view of mathematical point and many results were obtained. For example, the Cauchy problem for CH and periodic CH equation were studied in [6, 7, 8], the existence of global weak solutions and global conservative and dissipative solutions were established [9, 10]. The peakon solutions and smooth solitary wave solutions were obtained and proved to be orbital stable and they interact like solitons. Readers can refer to [3] for the details about wave breaking.

Special solutions play an important role in the research of partial differential equations, and it can be used to describe and explain many phenomena in physics and engineering and so on. It is interesting to consider the different kinds of exact
solutions of (1.1) and its generalized forms. Furthermore, some authors have studied a generalized CH equation as follows:

$$u_t + 2ku_x - u_{xx} + auu u_x = 2uu_x u_{xx} + uu_{xxx}$$  \hspace{1cm} (1.2)$$

they have considered the effect of the stronger nonlinear convection on the traveling waves, that is, the nonlinear term $uu u_x$ in (1.2) is used instead of the nonlinear convection term $uu u_x$ in (1.1), which changes the structure of the equation and leads to some new nonlinear phenomena arising from (1.2), such as compacton solitons with compact support, solitons with cusps, or peakons (see [11, 12, 13, 14, 15, 16]). Here we briefly recall some results about the solutions of (1.2). In [11], four simple ansätze were introduced to obtain abundant solutions: compactons, solitary patterns solutions having infinite slopes or cusps, and solitary waves. By applying bifurcation method, peakons and periodic cusp waves for (1.2) were studied in [12, 13, 14]; the exact expressions of peakons were obtained in some special cases. Several new exact peaked solitary waves were derived in [15]. By using polynomial ansätze the periodic wave and peaked solitary waves of (1.2) were investigated in [16]. The local existence for the Cauchy problem of (1.2) was established in [17, 18].

As far as we know, it is not complete although some exact solutions have been obtained for (1.2). There are a lot to be investigated about special solutions. For example, traveling delta wave is a very singular wave, which has not been considered yet. In [19, 20, 21, 22, 23, 24], in order to deal with non-smooth or distribution solutions of some nonlinear partial differential equations, such as Delta function, Heaviside function etc., the author has constructed a very suitable definition of products of distribution so that the results remain distributions for any product of distributions. It is a reasonable and effective extension of products of classical functions or distribution multiplied by smooth function, and can return to the classical products if both the factors multiplied by each other are classical functions. We will introduce the details about the products of distributions later. It is worth noticing that, in [19, 20, 21, 22, 23, 24], only the first order partial differential models were studied. So far, the higher order partial differential equations, even the second order ones, have not been considered in this way yet. It is a new attempt to use these methods in the above references to study the distribution solutions for a third order equation like (1.2). So, in this paper, we will use the relative definition and approach on products of distributions therein to research some specific aspects of propagation of delta waves for (1.2). It is proved that, in a sense of products of distributions defined by [19, 20, 21, 22, 23, 24], under certain conditions, the traveling delta wave

$$u(x,t) = r \delta(x-ct),$$  \hspace{1cm} (1.3)$$

is a solution of (1.2), where $\delta$ stands for the Dirac measure concentrated at the origin.

First of all, it is necessary to give an overview on the products of distributions because we have to depend on such products of distributions to obtain the relative results. Non-smooth functions or singular functions can be regarded as distributions or generalized functions, we have to turn to distributions or generalized functions when we want to obtain non-smooth or singular solutions for nonlinear partial differential equations because of the nonlinearity. In this situation, and from 1977 on, several distribution algebras were first introduced by Maslov and his collaborators, and later on by Rosinger. These works brought brought into light algebraic structures involved in embedding the space of distributions $D_0$ into certain quotient algebras. An excellent guide, for a preliminary review about those type of approaches to products of distributions, is the article of Egorov. From 1982 on, several products of distributions arose; a most popular one being that of Colombeau, especially related to the framework of Rosinger. The book of Oberguggenberger is an excellent guide to this direction.

As is well known, unfortunately, some distributional products are probably not successful in multiplying distributions with a strong singularity at a given point such as, for instance, the product $\delta \delta$ of two Dirac-delta measures. Other approaches obtain such products at the price of leaving out the space of distributions. For example, $\delta \delta$ is an element of the Colombeau’s algebra $\mathcal{G}$, but this element has no associated distribution. Consequently, from the mathematical point of view, $\delta \delta$ is well defined but difficult to interpret at a level of theoretical physics; some indeterminacies also arise.

The approach in [19, 20] is a general theory that provides a distribution as the outcome of any product of distributions, once fix a certain function $\alpha$. Such a function quantifies the indeterminacy inherent to the products, and, once fixed, its physical interpretation becomes clear. They stress that this indeterminacy is not avoidable in general, and it plays an essential role in many questions. Concerning this point, we can refer to Section 6 in [20]. For instance, within their framework, they have exhibited explicitly [19, 23] Dirac delta wave solutions (and also solutions which are not measures) for the turbulent model ruled by Burgers nonconservative equation, and some phenomena like infinitely narrow soliton solutions, obtained by Maslov and his collaborators arise directly in distributional form [23] as a particular case. Also in the same setting, for a model ruled by a singular perturbation of Burgers conservative equation, they have proved [24] that delta-waves under collision behave as classical soliton collisions (as in the Korteweg-de Vries equation).

The rest of this paper is organized as follows. In section 2, we give a review about the Delta and Heaviside functions and
their some properties used later. And then we introduce the product of distributions in a particular sense and some arith-
metical rules in section 3. In section 4 we define the concept of $\alpha$-solution and show that it is a particular extension of
classical solution. Finally, under some conditions, we prove that (1.2) possesses traveling Delta wave and Heaviside wave
solutions in section 5.

2. Delta function and Heaviside function

2.1. Delta and Heaviside functions and their some properties

In mathematics, the Dirac delta function ($\delta$ function) is a generalized function or distribution. It is used to model the
density of an idealized point mass or point charge as a function equal to zero everywhere except for zero and whose integral
over the entire real line is equal to one.

That is,

$$
\delta(x) = \begin{cases} 
+\infty, & x = 0 \\
0, & x \neq 0
\end{cases}
$$

and which is also constrained to satisfy the identity

$$
\int^{-\infty}_{-\infty} \delta(x) \, dx = 1
$$

As there is no function that has these properties, the computations made by the theoretical physicists appeared to m-
thematicians as nonsense until the introduction of distributions by Laurent Schwartz to formalize and validate the co-
putations. As a distribution, the Dirac delta function is a linear functional that maps every function to its value at zero.

Here we present some properties will be used later. The delta function satisfies the following scaling property for a
non-zero scalar $\mu$

$$
\int^{-\infty}_{-\infty} \delta(\mu x) \, dx = \int^{+\infty}_{-\infty} \frac{dy}{|\mu|} = \frac{1}{|\mu|}
$$

and so

$$
\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}
$$

In particular, the delta function is an even distribution, in the sense that

$$
\delta(-x) = \delta(x)
$$

The distributional derivative of the Dirac delta distribution is the distribution $\delta'(x)$ defined on compactly supported
smooth test functions $\varphi$

$$
\delta'[\varphi] = -\delta[\varphi'] = -\varphi'(0)
$$

The above equality here is a kind of integration by parts, for if $\delta$ is a true function then

$$
\int^{+\infty}_{-\infty} \delta'(x) \varphi(x) \, dx = -\int^{+\infty}_{-\infty} \delta(x) \varphi'(x) \, dx
$$

The $k$-th derivative of $\delta$ is defined similarly as the distribution given on test functions by

$$
\delta^{(k)}[\varphi] = (-1)^k \varphi^{(k)}(0)
$$

In particular, $\delta$ is an infinitely differentiable distribution. Furthermore, the convolutions of $\delta$ and $\delta'$ with a com-
pactly supported smooth function $f$ are

$$
\delta * f = f * \delta = f
$$

and

$$
\delta' * f = \delta * f' = f'
$$

respectively, which follow from the properties of the distributional derivative of a convolution.
2.2. Heaviside function and its some properties

The Heaviside step function, or the unit step function, usually denoted by $H$, is a discontinuous function, whose value is zero for negative arguments and one for positive arguments. It is an example of the general class of step functions, all of which can be represented as linear combinations of translations of this one.

The function was originally developed in operational calculus for the solution of differential equations, where it represents a signal that switches on at a specified time and stays switched on indefinitely. Oliver Heaviside, who developed the operational calculus as a tool in the analysis of telegraphic communications, represented the function as $1$.

The simplest definition of the Heaviside function is as the derivative of the ramp function:

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$  \hspace{1cm} (2.2)

The Heaviside function can also be defined as the integral of the Dirac delta function. This is sometimes written as

$$H(x) = \int_{-\infty}^{x} \delta(s) \, ds,$$  \hspace{1cm} (2.3)

although this expansion may not hold (or even make sense) for $x = 0$, depending on which formalism one uses to give meaning to integrals involving $\delta$. The ramp function is the antiderivative of the Heaviside step function

$$\int_{-\infty}^{x} H(\xi) \, d\xi = xH(x)$$  \hspace{1cm} (2.4)

The distributional derivative of the Heaviside step function is the Dirac delta function

$$\frac{dH(x)}{dx} = \delta(x)$$  \hspace{1cm} (2.5)

3. Product of Distributions

This section introduces the product of distributions defined in [32, 33].

Let $D$ be the space of compactly supported infinitely differentiable complex-valued functions defined on $R$, let $D'$ be the space of Schwartz distributions, and let $\alpha \in D$ be even with $\int_{-\infty}^{\infty} \alpha = 1$. In the theory of products in [32, 33], for computing the $\alpha$-product $T_{\alpha}S$, they arrive at a relation of the form

$$T_{\alpha}S = T\beta + (T \ast \alpha)f$$  \hspace{1cm} (3.1)

for $T \in D'$ and $S = \beta + f \in C^P \oplus D'_\mu$, where $P \in \{0,1,2,\ldots,\infty\}$, $D'^p$ is the space of distributions of order $P$ in the sense of Schwartz ($D'^{\infty}$ means $D'$), $D'_\mu$ is the space of distributions whose support has measure zero in the sense of Lebesgue, and $T\beta$ is the usual Schwartz product of a $D'^p$ distribution by a $C^P$-function.

Let $D'^{-1}$ be the space of distributions $T \in D'$ whose distributional derivatives are in $D'^0$, we can see that $T \in D'$ is locally a function of bounded variation. Then the $\alpha$-product of the distribution $T \in D'^{-1}$ by the distribution $S = \omega + f \in L^1_{\text{loc}} \oplus D'_\mu$ can also be defined by the following formula

$$T_{\alpha}S = DTF - (DT)F + (T \ast \alpha)f,$$  \hspace{1cm} (3.2)

where $F \in C^0$ satisfies that $DF = \omega$, we refer to [32] for details. Note that the convolution $T \ast \alpha$ in 3.1 and 3.2 is not regarded as an approximation of $T$, formulas 3.1 and 3.2 are to be treated as precise ones.

For instance, if $\delta$ stands for the Dirac measure and $H$ is the Heaviside function, we have

$$\delta \ast \delta = \delta(0 + \delta) = (\delta \ast \alpha)\delta = \alpha \delta = \alpha(0)\delta,$$  \hspace{1cm} (3.3)

$$\delta \ast (D\delta) = (\delta \ast \alpha)(D\delta) = \alpha(0)(D\delta) - \alpha'(0)\delta = \alpha(0)(D\delta).$$  \hspace{1cm} (3.4)
\[ (D\delta)_a \delta = ((D\delta) \ast \alpha)\delta = (\delta \ast \alpha')\delta = \alpha'(0)\delta = 0, \tag{3.5} \]
\[ H_a \delta = (H \ast \alpha)\delta = \left[ \int_{-\infty}^{\infty} \alpha(x - y)H(y)dy \right] \delta = \left[ \int_{-\infty}^{\infty} \alpha(-y)H(y)dy \right] \delta = \frac{1}{2}\delta, \tag{3.6} \]
and
\[ H_{a}H = H, \quad H_{a}^{n} = H. \tag{3.7} \]

It is easy to define the product of a distribution with a smooth function. A limitation of the theory of distributions is that there is no associative product of two distributions extending the product of a distribution by a smooth function, as has been proved by Laurent Schwartz in the 1950s. So about the properties of this kind of product of distributions, it is quite different from the pointwise product of classical functions.

This \( \alpha \)-product is bilinear, has unit element (the constant function taking the value 1 viewed as a distribution), is invariant under translations and also under the action of the transformation \( t \to -t \) from \( \mathbb{R} \) onto \( \mathbb{R} \). In general, this product is neither associative nor commutative, however,
\[ \int_{a} T_{a}S = \int_{R} S_{a}T \tag{3.8} \]
for any \( \alpha \), if \( T, S \in D'_{\mu} \) and \( T \) or \( S \) is compactly supported. In general, \( \alpha \)-products cannot be completely localized. This becomes clear by noticing that \( \text{supp}(T_{a}S) \subseteq \text{supp} S \) (as for ordinary functions), but it can happen that \( \text{supp}(T_{a}S) \subseteq \text{supp} T \). Thus, in the following, \( \alpha \)-product is regarded as global product, and when we apply product to differential equations, the solutions are naturally viewed as global solutions. The product (3.1) and (3.2) are consistent with the Schwartz product of \( D^{\prime \prime p} \)-distributions by \( C^{p} \)-functions (if these functions are placed on the right-hand side) and satisfy the standard differential rules.

In general, \( \alpha \)-product cannot be completely localized. Thus, in the following, \( \alpha \)-product is regarded as global product. The Leibniz formula must be represented in the form
\[ D(T_{a}S) = (DT)_{a}S + T_{a}(DS), \tag{3.9} \]
where \( D \) is the derivative operator in the distributional sense.

Besides, we can use \( \alpha \)-products (3.1) to define powers of some distributions. Thus, if \( T = \beta + f \in C^{\prime} \oplus D^{\prime}_{\mu} \oplus D^{\prime}_{p} \), then
\[ T_{a}T = \beta^{a} + \left[ \beta + (\beta \ast \alpha) + (f \ast \alpha) \right] f, \tag{3.10} \]
because \( T \in D^{\prime}_{\mu} \cap (C^{\prime} \oplus D^{\prime}_{p}) \). Since \( T_{a}T \in C^{\prime} \oplus D^{\prime}_{p} \cap D^{\prime}_{p} \), we can define the \( \alpha \)-powers \( T_{a}^{n} \) (\( n \geq 0 \) is an integer) by the recurrence formula
\[ T_{a}^{0} = 1, \tag{3.11} \]
\[ T_{a}^{n} = (T_{a}^{n-1})_{a}T. \tag{3.12} \]

Since the distributional products (3.1) are consistent with the Schwartz products of distributions by functions (when functions are placed on the right-hand side), we have \( \beta_{a}^{n} = \beta^{n} \) for all \( \beta \in C^{p} \), and the consistence of this definition with the ordinary powers of \( C^{p} \)-functions is proved. For instance, if \( r \in C \), then \( (r\delta)_{a}^{0} = 1 \) and \( (r\delta)_{a}^{n} = r^{n}[(\alpha(0))^{n-1} \) for \( n > 2 \), which can readily be seen by induction. We also have \( (rT)_{a}^{n} = r_{a}(T_{a}^{n}) \) in the distributional sense, where \( r_{a} \) is the translation operator defined by \( s \in \mathbb{R} \). Thus, in what follows, we shall write \( T^{n} \) instead of \( T_{a}^{n} \) (supposing that \( a \) is fixed), which will also simplify the notation.

Notice that, under the definition of this kind product of distributions, if \( \phi(u) \) is an entire function of \( u \), then \( \phi \circ u \) is well defined, here \( \phi \circ u \) is used to denotes the expression of \( \phi(u) \) involving the product of distributions, and we have the following result.
Lemma 3.1. [33] If \( \phi(u) \) is an entire function of \( u \), then
\[
\phi \circ (m_0) = \begin{cases} 
\phi(0) + \phi'(0)m_0\delta & \text{if } \alpha(0) \neq 0, \\
\phi(0) + \frac{\partial \phi[m_0\alpha(0)]}{\partial \alpha(0)} - \phi(0) \delta & \text{if } \alpha(0) \neq 0.
\end{cases}
\]

(3.13)

Proof. If \( \phi(u) \) is an entire function of \( u \), then we have
\[
\phi(u) = a_0 + a_1u + a_2u^2 + \ldots 
\]
where \( a_i = \frac{\phi^{(i)}(0)}{i!} \) for \( i = 0, 1, 2, \ldots \). For \( T \in D' \cap (C^p \otimes D'_p) \), we define the composition \( \phi \circ T \) as follows
\[
\phi \circ T = a_0 + a_1T + a_2T^2 + \ldots 
\]
provided this series converges in \( D' \). This is clearly a consistent definition, and we have \( \tau_\alpha (\phi T) = \phi \circ (\tau_\alpha T) \) if \( \phi \circ T \) or \( \phi \circ (\tau_\alpha T) \) are well defined. Recall that \( \phi \circ T \) depends on \( \alpha \) in general. Now we shall show that \( \phi \circ (r\delta) \) is a distribution for all \( r \in C \). We have \( (r\delta)^0 = 1 \) and \( (r\delta)^1 = r\delta \) and, for \( i \geq 2 \),
\[
(r\delta)^i = r'[\alpha(0)]^{-1} \delta
\]
(3.16)
as we have already seen. Then, according to (3.15),
\[
\phi \circ (r\delta) = a_0 + a_1r\delta + a_2(r\delta)^2 + \ldots 
\]
(3.17)because, as we shall see, this series is convergent in \( D' \). Indeed, by (3.16), we have
\[
\phi \circ (r\delta) = a_0 + a_1r\delta + a_2r^2[\alpha(0)]^2 \delta + a_3r^3[\alpha(0)]^3 \delta + \ldots 
\]
(3.18)and thus, if \( \alpha(0) = 0 \), then \( \phi \circ (r\delta) = a_0 + a_1r\delta \), while, if \( \alpha(0) \neq 0 \), then
\[
\alpha(0)[\phi \circ (r\delta) - a_0] = a_1[\alpha(0)]^2 \delta + a_3[\alpha(0)]^3 \delta + \ldots 
\]
(3.19)which is equivalent to
\[
\alpha(0)[\phi \circ (r\delta) - a_0] = a_1[\alpha(0)]^2 \delta + a_3[\alpha(0)]^3 \delta + \ldots
\]
(3.20)because, by (3.14), the series \{\ldots\} converges to \( \phi(r\alpha(0)) - a_0 \). In this case,
\[
\alpha(0)[\phi(r\alpha(0)) - a_0] = [\phi(r\alpha(0)) - a_0] \delta
\]
(3.21)from above equation, we have
\[
\phi \circ (r\delta) = \phi(0) + \frac{\phi(r\alpha(0)) - \phi(0)}{\alpha(0)} \delta
\]
(3.22)This completes the proof.

4. The Concept of \( \alpha \)-Solution

Now we introduce the concept of \( \alpha \)-Solution in this section. Let us consider equation (1.2). By a classical solution of (1.2) we mean a third order continuously differentiable complex function \( (x, t) \rightarrow u(x, t) \) which satisfies (1.2) at every point of its domain. Let \( I \) be an interval of \( R \) with nonempty interior and let \( F(I) \) be the space of second order continuously differentiable mappings \( \tilde{u} : I \rightarrow D' \) in the sense of the topology of \( D' \). For \( t \in I \), the notation \([\tilde{u}(t)](x)\) is sometimes used to stress that the distribution \( \tilde{u}(t) \) acts on functions \( \tilde{\xi} \in D \) that depend on \( x \).

Definition 4.1. The mapping \( \tilde{u} \in F(I) \) is said to be an \( \alpha \)-solution of (1.2) if and only if there is an \( \alpha \) such that, for all
\[ t \in I, \]
\[ (1 - D^{(2)}) \frac{d\tilde{u}(t)}{dt} + D[2k\tilde{u}(t) + \frac{a}{n + 1} \tilde{u}(t)^{n+1}] - 2D\tilde{u}(t)_{a} D^{(2)}\tilde{u}(t) - \tilde{u}(t)_{a} D^{(3)}\tilde{u}(t) = 0, \]  
\[ (4.1) \]
where \( D^{(n)}(n = 1,2,3) \) stand for the distributional derivatives.

**Theorem 4.1.** If \( u \) is a global classical solution of equation (1.2) on \( R \times I \) then, for any \( a \), the map \( \tilde{u} \) defined by \( \tilde{u}(t)(x) = u(x, t) \) is a global \( a \)-solution of (1.2).

**Theorem 4.2.** If \( u : R \times [0, +\infty) \rightarrow C \) a \( C^{3} \)-function and \( \tilde{u} : [0, +\infty) \rightarrow D' \) defined by \( \tilde{u}(t)(x) = u(x, t) \) is a global \( a \)-solution of (1.2), then \( u \) is a global classical solution of (1.2).

For the proof, it is sufficient to note that a \( C^{3} \)-function \( x \) can be treated as a continuously differentiable function \( x \) and to use the consistency of the \( a \)-products with the classical ones.

**Remark 4.1.** Theorems 4.1 and 4.2 show that \( a \)-solution is a particular extension of classical solution.

### 5. The propagation of a wave profile \( T \in D' \)

**Definition 5.1.** We call that \( T \in D' \) \( a \)-propagates with the speed \( c \), according to (1.2), if and only if the mapping \( \tilde{u} \in F(I) \) defined by \( \tilde{u}(t)(x) = u(x, t) \) and to use the consistency of the \( a \)-products with the classical ones.

**Theorem 5.1.** Let \( T \in D' \) be a nonconstant distribution. Then \( T \) \( a \)-propagates with the speed \( c \), according to (1.2), if and only if
\[ c(D - D^{(3)})T + D[2kT + \frac{a}{n + 1} \tau_{a}^{n+1}] - 2DT_{a} D^{(2)}T - T_{a} D^{(3)}T = 0. \]  
\[ (5.1) \]

**Proof.** Assume that \( T \) \( a \)-propagates with the speed \( c \). By Definition (4.1) and (5.1) we have
\[ (1 - D^{(2)}) \frac{d(\tau_{a}(T))}{dt} + D[2k(\tau_{a}(T)) + \frac{a}{n + 1}(\tau_{a}(T))^{n+1}] - 2D(\tau_{a}(T))_{a} D^{(2)}(\tau_{a}(T)) - (\tau_{a}(T))_{a} D^{(3)}(\tau_{a}(T)) = 0, \]  
\[ (5.2) \]
for all \( t \in I \). According to [14, p.648], we have \( \frac{d(\tau_{a}(T))}{dt} = cD(\tau_{a}(T)) \), so, the above equation can be rewritten as
\[ (1 - D^{(2)})D(\tau_{a}(T)) + D[2k(\tau_{a}(T)) + \frac{a}{n + 1}(\tau_{a}(T))^{n+1}] - 2D(\tau_{a}(T))_{a} D^{(2)}(\tau_{a}(T)) - (\tau_{a}(T))_{a} D^{(3)}(\tau_{a}(T)) = 0. \]  
\[ (5.3) \]
Using the translation operator \( \tau_{(\cdot)} \) to the above equation, we have
\[ c(D - D^{(3)})T + D[2kT + \frac{a}{n + 1} \tau_{(-\cdot)}^{n+1}] - 2DT_{a} D^{(2)}T - T_{a} D^{(3)}T = 0, \]  
\[ (5.4) \]
Since \( \tau_{(\cdot)}(T)_{a}^{n+1} = (\tau_{(\cdot)}(T))_{a}^{n+1} \), it follows that
\[ c(D - D^{(3)})T + D[2kT + \frac{a}{n + 1} \tau_{a}^{n+1}] - 2DT_{a} D^{(2)}T - T_{a} D^{(3)}T = 0. \]  
\[ (5.5) \]
Especially, if \( T = r\delta(x) \) or \( T = H(x) \) it can be verified that
\[ \frac{d(\tau_{(\cdot)}r\delta(x))}{dt} = -cD(\tau_{(\cdot)}r\delta(x)), \]  
\[ (5.6) \]
and
\[ \frac{d(\tau_{(\cdot)}H(x))}{dt} = -c(\tau_{(\cdot)}\delta(x)), \]  
\[ (5.7) \]
Let \( \xi \in D \) is a test function, in fact,
\[
\frac{d(r_{ct}r\delta(x))}{dt},\xi >= \frac{d(r\delta(x-ct))}{dt},\xi >
\]
\[
= \lim_{h \to 0} \frac{r\delta(x-c(t+h))-r\delta(x-ct)}{h},\xi >
\]
\[
= \lim_{h \to 0} \frac{1}{h} [r\delta(x-c(t+h)),\xi > - r\delta(x-ct),\xi >]
\]
\[
= \lim_{h \to 0} \frac{1}{h} [r\xi(c(t+h))-r\xi(ct)]
\]
\[
= cr\xi'(ct)
\]
\[
= cr < \delta(x-ct),\xi'(ct) >
\]
\[
= -cr < \delta'(x-ct),\xi(ct) >
\]
\[
= -cr\delta'(x-ct),\xi(ct) >.
\]
that is
\[
\frac{d(r_{ct}r\delta(x))}{dt} = -rc\delta'(x-ct) = -cD(r_{ct}r\delta(x)),
\]
where the prime of \( \delta \) is the distributional derivative. Similarly, we have
\[
\frac{d(H(x-c(t+h))-H(x-ct))}{dt},\xi >
\]
\[
= \lim_{h \to 0} \frac{H(x-c(t+h))-H(x-ct)}{h},\xi >
\]
\[
= \lim_{h \to 0} \frac{1}{h} [H(x-c(t+h)),\xi > - H(x-ct),\xi >]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \int_{x-c(t+h)}^{x} \xi'(x)dx - \int_{x-ct}^{x} \xi'(x)dx \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \int_{c(t+h)}^{x} \xi'(x)dx - \int_{ct}^{x} \xi'(x)dx \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \int_{x-ct}^{x} \xi'(x)dx \right]
\]
\[
= \delta(x-ct),\xi(0) >
\]
\[
= -c\delta(x-ct),\xi(0) >.
\]
namely,
\[
\frac{d(r_{ct}H(x))}{dt} = -c\delta(x-ct) = -c(r_{ct}r\delta(x)).
\]
Now we show that a Dirac delta wave \( T = r\delta(x)\alpha \) - propagates with speed \( c \) is a solution of (1.2), \( r \in C \) is a nonzero constant.

**Theorem 5.2.** Dirac delta wave \( T = r\delta(x)\alpha \) - propagates with speed \( c \) according to (1.2) if and only if the \( \alpha \) function should be chosen with \( \alpha(0) = -cr \) and the parameters \( a \) and \( c \) satisfy \( a = -\frac{(c+2k)(n+1)}{r^{2n}c^{n}} \).

**Proof.** According to the definition of product of distributions and Lemma 3.1, calculating directly, we have
\[
r\delta(x)^{n+1} = r^{n+1}[\alpha(0)]^{n} \delta(x),
\]
and
By using Theorem (5.1), substituting \( T = r \delta(x) \) into (5.1) with (5.12), we have

\[ crD \delta - crD^{(3)} \delta + 2krD \delta + \frac{a}{n+1} r^{n+1} [\alpha(0)]^{n} D \delta - \alpha(0) D^{(3)} \delta = 0, \]

that is

\[ \left( cr + 2kr + \frac{a}{n+1} r^{n+1} [\alpha(0)]^{n} \right) D \delta - (cr + \alpha(0)) D^{(3)} \delta = 0, \]

the above equation holds true if and only if \( \alpha(0) = -cr \) and \( a = \frac{(c+2k)(n+1)}{r^n c^r} \).

Similarly, we can prove that, under some conditions, a Heaviside wave \( T = r H(x) \) \( \alpha \)-propagates with speed \( c \) according to (1.2) if and only if the \( \alpha \) function should be chosen with \( \alpha(0) = 0 \) and the parameters \( c, a \) and \( k \) satisfy \( c = \frac{1}{2r} \) and \( k = -\frac{1}{4r} - \frac{a}{2n+2} r^n \).

**Theorem 5.3.** Heaviside wave \( T = r H(x) \) \( \alpha \)-propagates with speed \( c \) according to (1.2) if and only if the \( \alpha \) function should be chosen with \( \alpha(0) = 0 \) and the parameters \( c, a \) and \( k \) satisfy \( c = \frac{1}{2r} \) and \( k = -\frac{1}{4r} - \frac{a}{2n+2} r^n \).

**Proof.** Based on Theorem (5.1), substituting \( T = r H(x) \) into (5.1), we have

\[ crDH - crD^{(3)} H + 2krDH + \frac{a}{n+1} D(rH)^{n+1} - 2r^2 DH \alpha D^{(3)} H - H \alpha D^{(3)} H = 0, \]

since \( DH = \delta \) and \( H \alpha D^{(3)} H = H \alpha D^{(3)} \delta = \frac{1}{2} D^{(2)} \delta \), it follows that

\[ cr \delta - crD^{(3)} \delta + 2kr \delta + \frac{a}{n+1} r^{n+1} \delta - 2r^2 \delta \alpha D \delta - \frac{1}{2} D^{(2)} \delta = 0, \]

that is

\[ \left( cr + 2kr + \frac{a}{n+1} r^{n+1} \right) \delta - 2r^2 \alpha(0) D \delta - \left( cr - \frac{1}{2} \right) D^{(2)} \delta = 0, \]

the above equation holds if and only if \( c = \frac{1}{2r} \), \( \alpha(0) = 0 \) and \( k = -\frac{1}{4r} - \frac{a}{2n+2} r^n \).

**6. Conclusion**

Up to now, only first order nonlinear partial differential equations have been investigated with this kind of product of distributions. This paper has extended the application of such product of distributions into a higher order nonlinear partial differential equation, under some conditions, it is verified that Dirac delta function and Heaviside function with a translation at speed \( C \) are singular solutions of (1.2). The result of this paper shows that more higher order nonlinear models are able to be dealt with in this way.

**References**