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A Note on Barnette's Conjecture

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Abstract

In 1969, Barnette conjectured that every 3-connected cubic planar bipartite graph is Hamiltonian. We obtain two results to help under- stand Barnette's conjecture. The first result is inspired by the generation theorem of 3-connected cubic planar bipartite graphs, which is the work of Holton, Manvel and McKay. We define two operations called *VO* and *EOR* and prove that all graphs which are generated by the two operations from a unique graph of order 8 are Hamiltonian. We deduce also an equivalent description for the 3-connectivity of simple cubic plane bipartite graphs by the recent research hotspots of quasi spanning tree of faces. We prove that every simple cubic plane bipartite graph *G* is 3-connected if and only if the contraction graph *HR*(*G*) is 2-connected and 3-edge-connected, which meaning that if every 2-connected, 3-edge-connected planar graph of whose all vertex degrees are even and more than four has one quasi spanning tree of faces, then every 3-connected cubic planar bipartite graph is Hamiltonian.

Keywords

Barnette's conjecture, Hamiltonicity, connectivity

1. Introduction

Tait [1] conjectured that every 3-connected cubic planar graph is Hamiltonian in 1884; this came to be known as Tait's conjecture. Tutte [2] constructed a counterexample and conjectured that every 3-connected cubic bipartite graph is Hamiltonian in 1946, but this was shown to be false by the discovery of the Horton graph. Barnette [3] proposed a weakened combination of Tait's and Tutte's conjectures in 1969, stating that every 3-connected cubic planar bipartite graph is Hamiltonian.

Deftnition 1.1 (C3CBP and CBP). A C3CBP graph is a 3-connected cubic plane bipartite graph and a CBP graph is a simple cubic plane bipartite graph.

Barnette's Conjecture (version 1). *Every C3CBP graph is Hamiltonian.*

Remark. Graphs drawn in such a way that no two edges meet in a point other than a common end are called plane graphs; abstract graphs that can be drawn in this way are called planar. Since 3-connected planar graphs have a unique embedding in the plane [4], 3-connected cubic planar bipartite graphs can be replaced with C3CBP graphs in Barnette's conjecture (version 1), which raises noambiguity.

Although the truth of Barnette's conjecture remains unknown, many partial and related results have been obtained. For these results, one may refer to [5-14, 16-20] and the references therein. In particular, Goodey [5] proved that every C3CBP graphs in which all faces have four or six edges is Hamiltonian. Holton, Manvel and McKay [7] proved that all C3CBP graphs of order greater than 8 can be generated by two operations *VO* and *EO*, which will be defined in Section 2. Another result was obtained by Alt et al. [13]. It proved that if the dual graph of a C3CBP graph can be vertex-colored with colors red, green and blue such that every green-blue cycle contains a vertex of degree 4, then the C3CBP graph is Hamiltonian.

In this paper, we obtain two results to help understand Barnette's conjecture. We develop a sufficient condition for a C3CBP graph to be Hamiltonian and analyze the connectivity of C3CBP graphs.

The rest of the paper is organized as follows. In Section 2, we recall one theorem used further and prove a lemma. In Section 3, we prove that all graphs which are generated by two operations *VO* and *EOR* from a C3CBP*rbg* graph of order 8 are Hamiltonian. In Section 4, we deduce an equivalent description for the connectivity of C3CBP*rbg* graphs.

2. Preliminaries

In this section, we recall one theorem used further and prove a lemma that the C3CBP graph with the smallest order 8 is unique.

Deftnition 2.1 (edge-operation *EO*(*f*,*e*1 ,*e*2))**.** Let *G* be a CBP graph and *f* represent one face of the graph *G*. Choose two edges *e*1 and *e*2 on face *f* which have an ***odd*** distance along the boundary of face *f*. *G′* = *EO*(*f*,*e*1,*e*2)(*G*) b by subdividing both edges e1 and e2 twice and linking up the new vertices as shown in the left of Figure 1.

Deftnition 2.2 (vertex-operation *VO*(*v*))**.** Let *G* be a CBP graph and *v* represent one vertex of the graph *G*. *G′* = *VO*(*v*)(*G*) by replacing the vertex *v* with a subgraph as shown in the right of figure 1.

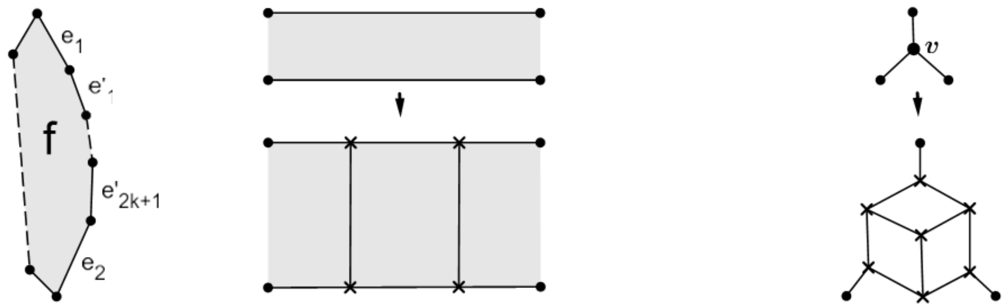

Figure 1. operations *EO* and *VO* on CBP.

Remark. We call the edge-operation *EO* and the vertex-operation *VO* for simplicity when causing no ambiguity.

Theorem 2.1 (Holton, Manvel and McKay [5]). *Let G be a C3CBP graph of order larger than* 8. *On the one hand, there always exits a C3CBP graph G′ satisfying either G* = *EO*(*G′*) *or G* = *VO*(*G′*). *On the other hand, both operations EO and VO preserve C3CBP property.*

Barnette's Conjecture (version 2). *All graphs which are generated by operations EO and VO from the C3CBP graph of order 8 are Hamiltonian.*

Lemma 2.2. *The C3CBP graph with the smallest order 8 is unique.* [5]

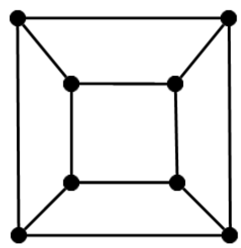

Figure 2. C3CBP graph of order 8.

Proof. Let *tt* with *n* vertices and *m* edges be a C3CBP graph with the smallest order. Considering *tt* is bipartite and regular, *n* is even. If *n* = 2, then *tt* is not simple. If *n* = 4, then *m* = 6 by 3*m* = 2*n*. Denote the face of length *k* by *k*-face and the number of *k*-faces by *fk*. By double counting and Euler's formula, we have that 4*f*4 = 2*m* and *n* + *f*4 − *m* = 2, which cause a contradiction that *f*4 = 6 and *f*4 = 4. If *n* = 6, by the same way, we have that *m* = 9, 4*f*4 + 6*f*6 = 2*m* and *n* + *f*4 + *f*6 − *m* = 2, which cause a contradiction that *f*4 = −3. Considering a C3CBP graph *G*0 of order 8 has been shown in Figure 2, *n* is equal to 8.

Next to prove *G* = *G*0. Since *n* = 8 and *m* = 12, we have that

$$4f4+6f6+8f8=24,\ f4,f6,f8\in \mathrm{N}$$

$$f4 + f6 + f8 - 4 = 2.$$

Because there exists only one integer solution that *f*4 = 6, *f*6 = *f*8 = 0, it is proved that *G* = *G*0. Thus, the C3CBP graph with the smallest order 8 is unique.

Remark. Though lemma 2.2 has been mentioned in the work by Holton, Manvel and McKay [5], they did not give a proof. Considering lemma 2.2 will be used a lot later, we give one proof.

3. A sufficient condition for C3CBP$_{rgb}$ graphs to be Hamiltonian

In this section, we define C3CBP$_{rbg}$ and CBP$_{rbg}$ graphs and prove that all C3CBP graphs *G* can be C3CBP$_{rgb}$. Then we define the edge-operation *EOR* and prove that all graphs which are generated by operations *EOR* and *VO* from the C3CBP$_{rgb}$ graph of order 8 are Hamiltonian. At last, we compare our results with some results obtained by others.

Deftnition 3.1 (C3CBP*rgb* and CBP*rgb*)**.** A C3CBP*rgb* graph is a C3CBP graph of which the faces are colored by 3 colors red, green and blue to make the adjacent faces have different colors. The same goes for CBP$_{rgb}$.

Theorem 3.1. *Every C3CBP graph G can be C3CBP$_{rgb}$.*

Proof. The C3CBP graph *G*0 of order 8 can be C3CBP$_{rgb}$ as shown in the left of Figure 4. Assume that all C3CBP graphs of order less than *n* can be C3CBP$_{rgb}$. Since all C3CBP graphs can be generated by *EO* and *VO* from graph *G*0, a C3CBP graph of order *n* is denoted by *G*1 and at least one of the follows can be true.

Case 1: *G*1 = *EO*(*f*,*e*1 ,*e*2)(*G*) and *G* is C3CBP. By assumption, C3CBP graph *G* can be C3CBP$_{rgb}$. Denote the color of face *f* by color *c*1 and the color of faces *fi* adjacent with *f* by color *c*2 and *c*3 alternatively. Thus there exists a face coloring of *G*1 which preserves the color of all faces but face *f* in *G* and changes the local coloring of face *f* in *G* as shown in the left of Figure3. It is promised by the odd distance between edges *e*1 and *e*2.

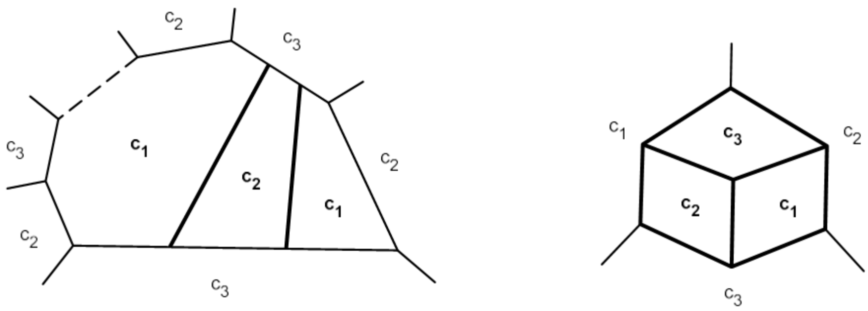

Figure 3. Face colouring after operations *EO* and *VO*.

Case 2: G_1 = *VO*(*v*)(*G*) and *G* is C_3CBP. By assumption, C_3CBP graph *G* can be C_3CBP$_{rgb}$. Thus there exists a face coloring of G_1 which preserves the color of all faces in graph *G* and color the three new faces in graph G_1 as shown in the right of Figure 3.

Thus, by induction on the graph order *n*, it is proved that all C_3CBP graphs *G* can be C_3CBP$_{rgb}$.

Remark. Since Heawood theorem [15] states that a plane triangulation *G* is 3-vertex-colorable if and only if the degree of each vertex in graph *G* is even, we can deduce that every CBP graph *G* can be CBP*rgb* by dual graphs. We still give another proof of theorem 3.1 by using theorem 2.1 because we want to show the changes of face coloring after operations *EO* and *VO*, which will be used later.

Figure 4. Left: the C_3CBP$_{rgb}$ graph *G* of order 8. Right: *HR*(*G*).

Deftnition3.2 (edge-*R*-operation *EOR*(*f*,*e*1,*e*2))**.** Let *G* bea CBP $_{rgb}$ graph. *EOR* (*f*,*e*1,*e*2) is an *EO* (*f*,*e*1,*e*2) operation which satisfies that the colors of faces *f* and *fi* which is adjacent with edge *ei* and face *f*are ***not*** red.

Remark. We call the edge-*R*-operation *EOR* for simplicity when causing no ambiguity. The same goes for the operation *EOG* when not green and the operation *EOB* when not blue.

BarneGe's Conjecture (version 3)**.** *All graphs which are generated by operations EOR, EOG, EOB and VO from the C3CBPrgb graph of order 8 are Hamiltonian.*

Theorem 3.2. *Let G be a C3CBPrgb graph of order n. If G is generated by operations EOR and VO from the*

C3CBPrgb graph of order 8, then G has a Hamiltonian cycle CH which contains all edges connecting two red faces.

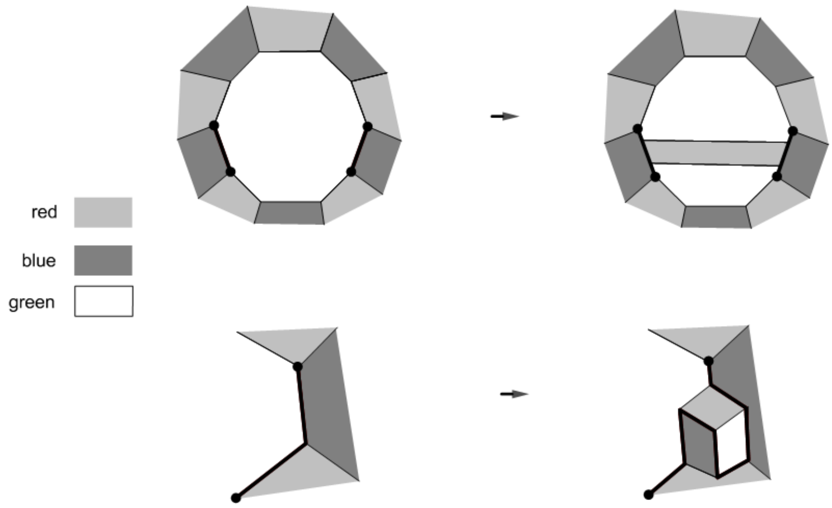

Figure 5. Operations *EOR* and *VO* on CBP$_{rgb}$.

Proof. If $n = 8$, then the C_3CBP_{rgb} graph *G* has a Hamiltonian cycle *CH* which contains all edges connecting two red faces as shown in the left of Figure 4 drawn with bold lines. Then, we make the proof by induction on *n*. Assume that if the order of a C_3CBP_{rbg} graph *G*0 is less than *n*, then the theorem 3.2 holds. Since the graph *G* is be generated by *EOR* and *VO*, at least one of the follows can be true.

Case 1: $G = EOR(f, e1, e2)(G')$ and G' is C3CBP*rgb*. By assumption, G' Has a Hamiltonian cycle CH' which contains all edges connecting two red faces. By the definition of edge-R-operation *ER*, both edges *e*1 and *e*2 are not on red faces. Since every edge *e* in G' is either on one red face or connecting two red faces, the Hamiltonian cycle CH' contains edges *e*1 and *e*2. As shown in the top of figure 5, the Hamiltonian cycle CH' of G' can be viewed as a Hamiltonian cycle of *G* which naturally contains all edges connecting two red faces in *G*.

Case 2: $G = VO(v)(G')$ and G' is C3CBP*rgb*. By assumption, G' has a Hamiltonian cycle *CH* 'which contains all edges connecting two red faces.

As shown in the boGom of figure 5, the Hamiltonian cycle CH' of G' can be extended into a Hamiltonian cycle *CH* of *G* which contains all edges connecting two red faces in *G*.

Remark (I)**.** Goodey [5] proved that all C3CBP graphs in which all faces have four or six edges are Hamiltonian. Their sufficient condition limits the length of faces. However, ours contains C3CBP graphs which have the face with $2k$ edges and the integer *k* can be arbitrarily large. Compared with the result obtained by Goodey, ours make some progress.

Remark (II)**.** Holton, Manvel and McKay [7] proved that all graphs which are generated by operations *VO* from the C3CBP graph of order 8 are Hamiltonian. However, we prove that all graphs which are generated by operations *EOR* and *VO* from the C3CBP*rgb* graph of order 8 are Hamiltonian. Thus, we develop the sufficient condition for a C3CBP graph to be Hamiltonian.

Remark (III)**.** Operations *EOR* and *VO* increase the number of red faces by exactly one and operations *EOG* and *EOB* increase the number of red faces by no more than one. It is hopeful to develop the sufficient condition further if choosing another kind of specific Hamiltonian cycles by doing induction on the number of red faces and using the similar method. Meanwhile, the way of choosing one kind of Hamiltonian cycles can be quite flexible. Because some equivalent but stronger statements of BarneGe's conjecture has been proved. For example, Kelmans [8] proved that BarneGe's conjecture is equivalent to a superficially stronger statement that for ev- ery two edges *e* and *f* on the same face of a C3CBP graph, there exists a Hamiltonian cycle that contains the edge *e* but does not contain the edge *f*.

Theorem 3.3 (Alt et al. [13]). *Let G be a C3CBP graph of order n. If the dual graph of G can be vertex-colored with colors red, green and blue such that every green-blue cycle contains a vertex of degree 4, then G is Hamiltonian.*

Remark. The C3CBP*rgb* graph *G* shown in figure 6 is generated by operations *EOR*, *EOB* and *VO* from the C3CBP*rgb* graph of order 8.

Theorem 3.3 obtained by Alt et al. [13] fails to work on the graph. The dual graph $G*$ of graph *G* contains a red-blue cycle drawn by dashedl ines and the red-blue cycle has no verex of degree 4. It means that the result obtained by Alt et al. cannot behave be Ger than ours for solving the BarneGe's conjecture (version 3). However, the theorem 3.3 is proved by showing that the vertices of the dual of every C3CBP graph can be partitioned into two subsets whose induced subgraphs are trees, which is proved to be equivalent with BarneGe's conjecture by Florek [12]. It means that our proof is much more simple than theirs.

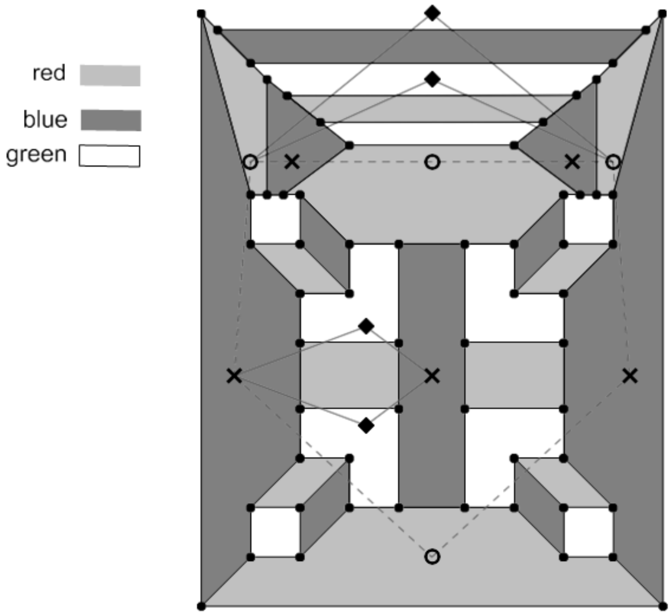

Figure 6. A counterexample.

4. 3-connectivity of C3CBP$_{rbg}$ graphs

In this section, we give some definitions and prove an equivalent statement for the 3-connectivity of C3CBP*rbg* graphs.

Deftnition 4.1 (*k*-*fR*-connected)**.** Let *G* be a CBP*rgb* graph. *G* is said to be *k*-*fR*-connected if it has more than *k* red faces and remains connected whenever fewer than *k* red faces are removed. Specifically, removing one face *f* means that removing all vertices on face *f*.

Deftnition 4.2 (*k*-*eRR*-connected)**.** Let *G* be a CBP*rgb* graph. *G* is said to be *k*-*eRR*-connected if it has more than *k* edges connecting two red faces and remains connected whenever fewer than *k* edges connecting two red faces are removed.

Lemma 4.1. *A cubic graph G has the property* $\kappa(G) = \kappa'(G)$.

Proof. Since $\kappa'(G) \geq \kappa(G)$, it is sufficient to construct an edge cut of size $\kappa(G)$. Denote the minimal vertex cut set of size $\kappa(G)$ by *S*. Because *G* is cubic and *S* is minimal, for each $v \in S$, there exists one component *C* in $G - S$ satisfying that the size of $V(C) \cap N(v)$ equals one and denote the vertex belongs to $V(C) \cap N(v)$ by wv . The edge set $E = \{wvv : v \in S\}$ of size $\kappa(G)$ is an edge cut set. Thus, $\kappa(G) = \kappa(G)$.

Deftnition 4.3 (contraction graph $H_R(G)$)**.** Let *G* be a CBP*rgb* graph. $H_R(G)$ is said to be a plane graph that contracts every red faces in graph *G* into a vertex and preserves the relative location of edges in graph *G*.

Remark. One simple example of $H_R(G)$ is shown in the right of figure 4.

Theorem 4.2. *Let G be a CBPrgb graph and the following statements are equivalent:*

1) *Gis3-connected.*
2) *G is 2-fR-connected and3-eRR-connected.*
3) *HR(G) is 2-connected and3-edge-connected.*

Remark. It is noticed that there exists 2-*fR*-connected CBP*rgb* graphs which are not 3-*eRR*-connected and 3-*eRR*-connected CBP*rgb* graphs which are not 2-*fR*-connected.

Proof. We can observe that every edge *e* in CBP*rgb* graph is either on one red face or connecting two redfaces.

Proof of 1⇒3: If the order of graph *G* is 8, $H_R(G)$ is 2-connected and 3-edge-connected as shown in right of figure 4. Then we make induction on the order of graph *G*. Denote the order of graph *G* by *n*. Assume that if the order of a C3CBP*rgb* graph G_0 is less than n, $H_R(G_0)$ is 2-connected and 3-edge-connected. By theorem 2.1, at least one of the follows can be true.

Case 1: $G = EO(f, e1, e2)(G')$ and G' is C3CBP*rgb*. Denote the color of face *f* by *cf* and the color of face *fi* which adjacent with face *f* and edge *ei* by *cei*. Since *ce*1=*ce*2, let *ce* be *cei*.

Case 1.1: *ce* is red.

As shown in the left of the second row of figure 7, the operation *EO* on graph G' will cause two vertices of $HR(G')$ which are on the same face linked up by two new edges. By assumption, $HR(G')$ is 2-connected and 3-edge-connected. Thus, $HR(G)$ is still 2-connected and 3-edge-connected.

Case1.2: both *ce* and *cf* are not red.

As shown in the middle of the second row of figure 7, the operation *EO* on graph G' will cause two edges $v1v2$ and $v3v4$ in $HR(G')$ which are on the face f_0 removed and a new vertex *w* added in the face f_0 with four new edges wv_1, wv_2, wv_3 and wv_4. By assumption, $HR(G')$ is 2-connected and 3-edge-connected.

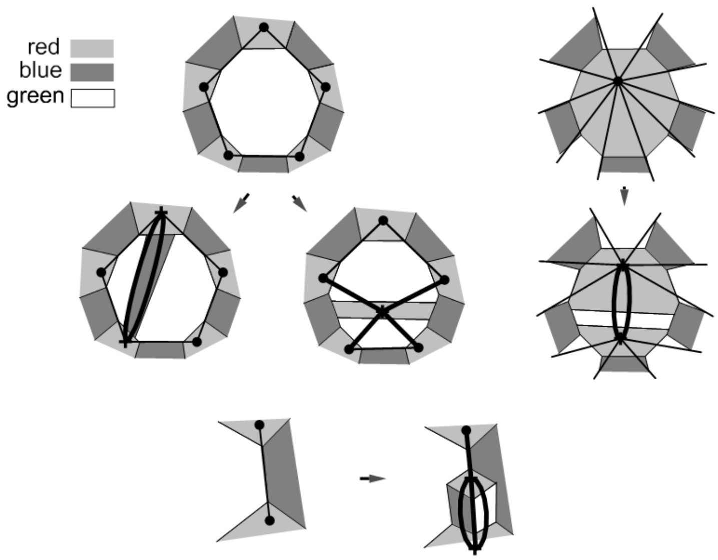

Figure 7. Operations *EO* and *VO* on $HR(G)$.

If $HR(G)$ is not 2-connected, then $HR(G)$ has a cut vertex v_0. If $v_0 = w$, then the vertex v_0 can also be the cut vertex of $HR(G')$ which is contradict with 2-connectivity. If $v_0 = w$, then $S = \{v_1v_2, v_3v_4\}$ can be the edge cut set of $HR(G)$ which is contradict with 3-edge-connectivity. Thus, $HR(G)$ is 2-connected.

If $HR(G)$ is not 3-edge-connected, then $HR(G)$ has an edge cut set S_0 of size 2. If S_0 is minimal, then the size of $S_0 \cap \{wv_1, wv_2, wv_3, wv_4\}$ can not equal one. If $S_0 \cap \{wv_1, wv_2, wv_3, wv_4\} = \emptyset$, then S_0 can also be the edge cut set of $HR(G)$ which is contradict with 3-edge-connectivity. If $S_0 \cap \{wv_1, wv_2, wv_3, wv_4\} = S_0$, then the vertex *w* can be the cut vertex of $HR(G)$ which is contradict with 2-connectivity. If S_0 is not minimal, then at least one edge of S_0 can be the bridge of $HR(G)$ and denote the edge by u_1u_2. The vertex u_1 belonging to $HR(G)$ can be the cut vertex of $HR(G)$ which is contradict with 2-connectivity. Thus $HR(G)$ is3-edge-connected.

Case1.3: *cf* is red.

As shown in the right of the second row of figure 7, the operation *EO* on graph G' will cause one vertex *VO*f $HR(G')$ split into two vertices $v1$ and $v2$. Denote the length of face *f* by $l(f)$ and set $l(f)$ equal $2k$. Let the edges adjacent with the vertex *v* indexed by $w1, w2, .., w2k$ clockwise. Part of $\{wi\}$ with consecutive index will be adjacent with the vertex $v1$ and the remained part will be adjacent with the vertex $v2$. The vertices $v1$ and $v2$ are also linked up by two new edges *ea* and *eb* and the degree of vertex *vi* is no less than 4. By assumption, $HR(G')$ is 2-connected and3-edge-connected.

If $HR(G)$ is not 2-connected, then $HR(G)$ has a cut vertex $v0$. If $v0 \notin \{v1, v2\}$, then the vertex $v0$ can also be the cut vertex of $HR(G')$ which is contradict with 2-connectivity. If $v0 \in \{v1, v2\}$, then the vertex *v* can be the cut vertex of $HR(G)$ which is contradict with 2-connectivity. Thus, $HR(G)$ is 2-connected.

If $HR(G)$ is not 3-edge-connected, then $HR(G)$ has an edge cut set $S0$ of size 2. If $|S0 \cap \{ea, eb\}| = 0$, then $S0$ can also be the edge cut set of $HR(G')$ which is contradict with 3-edge-connectivity. If $|S0 \cap \{ea, eb\}| = 1$, then $S0$ is not minimal. Let $e = u1u2$ be the edge in the set $S = S0 - \{ea, eb\}$. The vertex $u1$ belonging to $HR(G)$ can be the cut vertex of $HR(G)$ which is contradict with 2-connectivity. If $|S0 \cap \{ea, eb\}| = 2$, then the vertex *v* can be the cut vertex of $HR(G)$ which is contradict with 2-connectivity. Thus $HR(G)$ is 3-edge-connected.

Case 2: $G = VO(v)(G')$ and G' is C3CBP*rgb*. As shown in the third row of figure 7, The operation *VO* on G' will cause one edge of $H_R(G')$ replaced by a subgraph. By assumption, $H_R(G')$ is 2-connected and 3-edge- connected. Thus, $H_R(G)$ is still 2-connected and 3-edge-connected.

Proof of 3⇒1: If graph G is not 3-connected, then, by lemma 4.1, there exists an edge cut set S of graph G and $1 \leq |S| \leq 2$.

Case 1: $|S| = 1$.

Let $S = \{e\}$. If the edge e connects two red faces, then the edge e will be the bridge of $H_R(G)$ which is contradict with the 3-edge-connectivity. If not, then the edge e is on one red face. Denote the red face by f. The corresponding vertex v_R of face f in $H_R(G)$ can be the cut vertex of $H_R(G)$ which is contradict with the 2-connectivity.

Case 2: $|S| = 2$.

Let $S = \{e_1, e_2\}$. Denote the two faces adjacent with edges e_i by f_{i1} and f_{i2}. Since S is the edge cut set of graph G, $\{f_{11}, f_{12}\} = \{f_{21}, f_{22}\}$. If edge e_1 connects two red faces, then the same goes for edge e_2. The corresponding edges e_i in $H_R(G)$ can be the edge cut of $H_R(G)$ which is contradict with the 3-edge-connectivity. If edge e_1 is on one red face, then the same goes for edge e_2. Since either face f_{11} or face f_{12} is red, without loss of generality, let face f_{11} be red. The corresponding vertex v_R of face f_{11} in $H_R(G)$ can be the cut vertex of $H_R(G)$ which is contradict with the 2-connectivity.

Proof of 1⇔2: By definition 4.1, the 2-f_R-connectivity of CBP *rgb* graph G is equivalent with the 2-connectity of the graph $H_R(G)$. By definition 4.2, the 3-e_{RR}-connectivity of CBP *rgb* graph G is equivalent with the 3-edge-connectity of the graph $H_R(G)$.

Remark. We have a deeper look into the connectivity of C3CBP*rgb* graphs. We hope this equivalent statement can help understand BarneGe's conjecture beGer and help develop the sufficient condition in Section 3 further. Besides, with the help of theorem 4.2, it is hopeful to design an algorithm to test whether a CBP *rgb* graph G of order n is 3-connected more efficiently.

Acknowledgement

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References

[1] Tait, P. G., *Listing's Topologie*, Philosophical Magazine, 5th Series, 1884, 17: 30–46.

[2] TuGe, W. T., *On Hamiltonian circuits*, Journal of the London Mathematical Society, 1946, 21 (2): 98–101.

[3] David W. BarneGe, Conjecture 5, Recent progress in combinatorics, Academic Press, New York, 1969, 343.

[4] H. Whitney, *Coqcruent graphs and the connectivity of graphs*, Am. J. Math., 1932,54 150-168.

[5] Goodey, P. R., *Hamiltonian circuits in polytopes with even sided faces*, Israel Journal of Mathematics, 1975, 22(1):52–56.

[6] Akiyama, Takanori; Nishizeki, Takao; Saito, Nobuji, *NP-completeness of the Hamiltonian cycle problem for bipartite graphs*, Journal of Information Processing, 1980, 3 (2): 73–76.

[7] Holton, D. A.; Manvel, B.; McKay, B. D., *Hamiltonian cycles in cubic 3-connected bipartite planar graphs*, Journal of Combinatorial Theory, Series B, 1985, 38 (3): 279–297.

[8] Kelmans, A. K., *Constructions of cubic bipartite 3-connected graphs without Hamiltonian cycles*, American Mathematical Society Translations, 1994, 158 (2): 127–140.

[9] Aldred, R. E. L.; Bau, S.; Holton, D. A.; McKay, Brendan D., *Nonhamiltonian 3-connected cubic planar graphs*, SIAM Journal on Discrete Mathematics, 2000, 13 (1): 25–32.

[10] Hertel A. *A survey & strengthening of BarneGe's conjecture*, University of Toronto, 2005.

[11] Feder, Tomas; Subi, Carlos, *On BarneGe's conjecture*, ECCC, 2006.

[12] Florek, Jan, *On BarneGe's conjecture*, Discrete Mathematics, 2010, 310 (10–11): 1531–1535.

[13] Alt, Helmut; Payne, Michael S.; Schmidt, Jens M.; Wood, David R., *Thoughts on BarneGe's conjecture*, Australasian Journal of Combinatorics, 2016, 64 (2): 354–365.

[14] Kardoˇs, F., *Acomputer-assisted proof of the BarneGe-Goodey Conjecture: not only fullerene graphs are hamiltonian*, SIAM Journal on Discrete Mathematics, 2020, 34 (1): 62–100.

[15] P. J. Heawood. *On the four-colour map theorem*, Quart. J. Pure Appl. Math., 1898, 29: 270–285.

[16] Bagheri Gh B, Feder T, Fleischner H, et al. Hamiltonian cycles in planar cubic graphs with facial 2 - factors, and a new partial solution of Barnette's Conjecture [J]. Journal of Graph Theory, 2021, 96(2): 269-288.

[17] Florek J. Graphs with multi-$4 $-cycles and the Barnette's conjecture [J]. arXiv preprint arXiv:2002.05288, 2020.

[18] Gorsky M, Steiner R, Wiederrecht S. Matching theory and Barnette's conjecture [J]. Discrete Mathematics, 2023, 346(2): 113249.

[19] Gh B B, Feder T, Fleischner H, et al. On finding hamiltonian cycles in Barnette graphs [J]. Fundamenta Informaticae, 2022, 188.

[20] Bej S. Hamiltonian cycles in annular decomposable Barnette graphs [J]. Journal of Discrete Mathematical Sciences and Cryptography, 2022: 1-15.