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Joint Moments of Surplus Immediately Before Ruin and Deficit at Ruin in the Renewal Risk Model

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Abstract

This paper studies the risk process where the individual claim amount density and the interclaim time density are assumed to be arbitrary. It uses the renewability of the risk process at the time of claims and the cumulative law of mathematical expectation to introduce a defective renewal equation satisfied by the joint moments of the surplus immediately before ruin and the deficit at ruin. From this equation, an explicit expression for the above joint moments (including the ruin probability) is obtained by the Laplace transform. Explicit expressions for the ruin probability are given for the case where the claim amount distribution is exponential, and numerical examples are given to analyze the effect of the relevant parameters on the ruin probability under the conditions that the interclaim times follow a Gamma distribution and a generalized Gamma distribution, respectively.

Keywords

Renewal risk model, Joint moment, Defective renewal equation, Laplace transform

1. Introduction

In the classical composite Poisson risk model, the interclaim time random variables are assumed to be independent and identically distributed with an exponential distribution function. Ref. [1] provided a comprehensive overview of the history and development of classical risk theory, and Ref. [2] investigated the joint moments of discounted claims and discounted perturbation until ruin in the compound Poisson risk model with diffusion. The renewal risk model (also known as the Sparre Andersen risk model) is a substantial generalization of the composite Poisson risk model that allows both the interclaim time and the claim amount to follow a more general distribution. Research progress on the renewal risk model can be found in [3-6], which discussed the renewal risk model in which the interclaim time or the claim amount obeyed some special distribution (e.g., a Gamma distribution).

Under the composite Poisson risk model, Ref. [7] introduced the expected discounted penalty function (referred to as the Gerber-Shiu function), which is used to analyze other functions related to the ruin event. Since then, the Gerber-Shiu function has become a unified tool for determining the quantities associated with ruin, and a large number of ruin theory studies have been devoted to the Gerber-Shiu function, which has been widely studied and extended to different surplus processes (see, e.g., [8-11]). The renewal risk model allows for a more general distribution of the interclaim time, and several researchers have contributed to the analysis of the Sparre Andersen risk model by studying the Gerber-Shiu function (see, e.g., [9, 10]).

In this paper, we generalize the risk model in which the interclaim times obey a specific distribution such as the exponential distribution to a renewal risk model with an arbitrary distribution of the interclaim times and an arbitrary distribution of the claim size. In this model, the Gerber-Shiu function is used as a tool to introduce a defective renewal equation satisfied

by the joint moments of the surplus immediately before ruin and the deficit at ruin by applying the law of iterated expectations and the surplus process renewability at the time of claims, without considering the discount factor. The Laplace transform and inverse transform are performed on both sides of this equation to obtain an analytic expression for the above joint moments, which in turn leads to the analytic expression for the ruin probability. An explicit expression for the ruin probability is given for the case where the claim size distribution is exponential. For exponentially distributed claim sizes, numerical examples are given for the two cases where the claims counting process is an Erlang(n) process and a generalized Erlang(2) process, respectively, and the effect of the relevant parameters on the ruin probability is analyzed.

2. The model and preliminaries

Consider an insurer's surplus process which is defined by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i , \tag{1}$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the constant premium rate per unit time, and N_t represents the number of claims over the time interval $(0,t]$. The claim number process $\{N_t, t \geq 0\}$ is assumed to be a renewal process defined by a sequence of positive, independent and identically distributed interclaim times $\{T_i, i = 1,2,\cdots\}$. That is, T_1 is the time of the first claim and $T_i, i = 2,3,\cdots$, is the time between the $(i-1)$th and ith claim. For an arbitrary $T_i, i = 1,2,\cdots$, define the interclaim time distribution function to be $K(t) = 1 - \bar{K}(t) = \Pr(T_i \leq t)$ and its probability density function to be 为 $k(t) = K'(t)$ for $t > 0$. Also, we assume $\{X_i, i = 1,2,\cdots\}$ is an independent and identically distributed sequence of positive claim size random variables and is independent of $\{T_i, i = 1,2,\cdots\}$. For an arbitrary $X_i, i = 1,2,\cdots$, let $F(x) = \Pr(X_i \leq x)$ be its distribution function and $f(x) = F'(x)$ for $x > 0$ be its probability density function. Besides, the *security loading condition* of $c\mathrm{E}(T_i) > \mathrm{E}(X_i)$ is imposed on the model so that ruin does not occur almost surely.

For the surplus process $\{U_t, t \geq 0\}$, we define the time of ruin as $T = \inf\{t \geq 0 : U_t < 0\}$ with the convention that $\inf \varnothing = \infty$, the time of the first drop in surplus below the initial level u as $T_u = \inf\{t \geq 0 : U_t < u\}$, and the ruin probability as $\psi(u) = \Pr(T < \infty | U_0 = u)$, respectively. The joint lth moment of the surplus immediately prior to ruin U_{T-} and mth moment of the deficit at ruin $|U_T|$ are defined by

$$\phi_{l,m}(u) = \mathrm{E}\left[(U_{T-})^l (|U_T|)^m I(T < \infty) | U_0 = u \right], \tag{2}$$

where both of l and m are non-negative integers, and $I(A)$ is the indicator function of the event A. See Figure 1 in [7] for U_{T-}, $|U_T|$, and a sample path of the renewal risk model (1).

3. A renewal equation for the joint moments

First, for the given initial surplus u, consider the conditional probability density function of $|U_T|$ at y, given that $U_{T-} = x$ and $T = t$, respectively. From the conditional probability formula, the conditional probability density function does not depend on t and is

$$f_x(y) = f(x+y) / (1 - F(x)), \ y \geq 0.$$

For the case where the first claim causes ruin, the joint probability density function of U_{T-}, $|U_T|$ and T at the point (x,y,t) is $k(t) f(x+y)$, where $t > 0, x = u + ct > u, y > 0$. Thus, the joint probability density function of U_{T-} and $|U_T|$ at the point (x,y) is

$$p_1(x,y|u) = \frac{1}{c} k\left(\frac{x-u}{c} \right) f(x+y) = \frac{1}{c} k\left(\frac{x-u}{c} \right) (1 - F(x)) f_x(y).$$

For the case where the first claim does not cause ruin, the joint probability density function of U_{T-}, $|U_T|$ and T at the point (x,y,t) is denoted as $p_2(t,x,y|u)$, where $t>0,\ 0<x<u+ct,\ y>0$. Thus, the joint probability density function of U_{T-} and $|U_T|$ at the point (x,y) is $p_2(x,y|u)=\int_0^\infty p_2(t,x,y|u)dt$.

Given $U_0=u$, from the above, it follows that the joint probability density function of U_{T-} and $|U_T|$ at the point (x,y) is

$$p(x,y|u)=p_1(x,y|u)+p_2(x,y|u)=f_{U_{T-}}(x|u)f_x(y)=f_{U_{T-}}(x|u)f(x+y)/(1-F(x)),$$

where $f_{U_{T-}}(x|u)$ is the probability density function of U_{T-}.

Proposition 1. *The joint moments $\phi_{l,m}(u)$ satisfies the following defective renewal equation*

$$\phi_{l,m}(u)=\alpha\int_0^u\phi_{l,m}(u-y)h(y)dy+\nu_{l,m}(u),\quad u\geq 0, \tag{3}$$

where

$$\alpha=\int_0^\infty\int_0^\infty p(x,y|0)dydx, \tag{4}$$

$$h(y)=\frac{1}{\alpha}\int_0^\infty p(x,y|0)dx, \tag{5}$$

$$\nu_{l,m}(u)=\int_u^\infty\int_0^\infty(u+x)^l(y-u)^m\,p(x,y|0)dxdy. \tag{6}$$

Proof. Let's now consider the following four cases: (i) $T_u=T_1$, and the first claim causes ruin. (ii) $T_u=T_1$, and the first claim does not cause ruin. (iii) $T_u>T_1$, and ruin occurs at the time T_u. (iv) $T_u>T_1$, and ruin does not occur at the time T_u. Given that the risk process (1) renews at claim instants, it follows that by the cumulative law of mathematical expectation

$$\begin{aligned}\phi_{l,m}(u)&=\int_0^u\phi_{l,m}(u-y)\left(\int_0^\infty p_1(x,y|0)dx\right)dy+\int_u^\infty\int_0^\infty(u+x)^l(y-u)^m\,p_1(x,y|0)dxdy\\&\quad+\int_0^u\phi_{l,m}(u-y)\left(\int_0^\infty p_2(x,y|0)dx\right)dy+\int_u^\infty\int_0^\infty(u+x)^l(y-u)^m\,p_2(x,y|0)dxdy\\&=\int_0^u\phi_{l,m}(u-y)\left(\int_0^\infty p(x,y|0)dx\right)dy+\int_u^\infty\int_0^\infty(u+x)^l(y-u)^m\,p(x,y|0)dxdy.\end{aligned} \tag{7}$$

From (4), (5), (6) and (7), it is observed that the equation (3) holds.

Similarly, applying the cumulative law of mathematical expectation yields

$$\phi_{l,m}(u)=\int_0^\infty\int_0^\infty x^l y^m\left(p_1(x,y|u)+p_2(x,y|u)\right)dxdy=\int_0^\infty\int_0^\infty x^l y^m p(x,y|u)dxdy,$$

By (2), (4) and the above equation, we have

$$0<\alpha=\phi_{0,0}(0)=\mathrm{E}\left[I(T<\infty)|U_0=0\right]=\Pr(T<\infty|U_0=0)=\psi(0), \tag{8}$$

From the *security loading condition* of $c\mathrm{E}(T_i)>\mathrm{E}(X_i)$, we get $\psi(u)<1$ (see Section 3.3 of [1] for more details) and $\alpha\in(0,1)$. Then, by (4) and (5), we have $\int_0^\infty h(y)dy=1$. Therefore, Eq. (3) is a *defective renewal equation* satisfied by $\phi_{l,m}(u)$. □

In the rest of this section, we derive explicit expressions for the joint moments $\phi_{l,m}(u)$. We denote the Laplace transform of an arbitrary function $\eta(x)$, $x\geq 0$, by $\hat{\eta}(s)=\int_0^\infty e^{-sx}\eta(x)dx$. Taking the Laplace transform of Eq. (3), one deduces that

$$\hat{\phi}_{l,m}(s)=\alpha\hat{\phi}_{l,m}(s)\hat{h}(s)+\hat{v}_{l,m}(s),$$

$$\hat{\phi}_{l,m}(s)=\frac{\hat{v}_{l,m}(s)}{1-\alpha\hat{h}(s)}=\hat{v}_{l,m}(s)\sum_{n=0}^{\infty}\left(\alpha\hat{h}(s)\right)^n=\frac{\hat{v}_{l,m}(s)}{1-\alpha}\left(1-\alpha+\sum_{n=1}^{\infty}(1-\alpha)\alpha^n\left(\hat{h}(s)\right)^n\right). \tag{9}$$

We construct two auxiliary functions as follows:

$$g(y)=\sum_{n=1}^{\infty}(1-\alpha)\alpha^n h^{*n}(y), \tag{10}$$

$$G(y)=1-\bar{G}(y)=1-\alpha+\int_0^y g(x)dx,\quad y\geq 0, \tag{11}$$

where $h^{*n}(y)$ is the n-fold convolution of $h(y)$ with itself. By $\int_0^{\infty}h(y)dy=1$, we have $\int_0^{\infty}h^{*n}(y)dy=1$, and

$$\int_0^{\infty}g(y)dy=\sum_{n=1}^{\infty}(1-\alpha)\alpha^n=\alpha\in(0,1).$$

Taking the Laplace transform of Eq. (10), we get that

$$\hat{g}(s)=\sum_{n=1}^{\infty}(1-\alpha)\alpha^n\left(\hat{h}(s)\right)^n, \tag{12}$$

$$1-\alpha\hat{h}(s)=\frac{1-\alpha}{1-\alpha+\hat{g}(s)}, \tag{13}$$

From (9) and (12) or (13), it follows that

$$\hat{\phi}_{l,m}(s)=\frac{\hat{v}_{l,m}(s)}{1-\alpha}\left(1-\alpha+\hat{g}(s)\right)=\hat{v}_{l,m}(s)+\frac{\hat{v}_{l,m}(s)\hat{g}(s)}{1-\alpha},$$

which we invert to obtain

$$\phi_{l,m}(u)=v_{l,m}(u)+\frac{1}{1-\alpha}\int_0^u g(u-y)v_{l,m}(y)dy. \tag{14}$$

Using integration by parts to the integral term on the right-hand side of (14), the explicit expression for the joint moments $\phi_{l,m}(u)$ becomes

$$\phi_{l,m}(u)=\frac{1}{1-\alpha}\left(v_{l,m}(u)-v_{l,m}(0)\bar{G}(u)-\int_0^u\bar{G}(u-y)v'_{l,m}(y)dy\right).$$

Next, we give explicit expressions for the ruin probability $\psi(u)$. From $\psi(u)=\mathrm{E}\left[I(T<\infty)\middle|U_0=u\right]=\phi_{0,0}(u)$, we note that

$$\psi(u)=v_{0,0}(u)+\frac{1}{1-\alpha}\int_0^u g(u-y)v_{0,0}(y)dy, \tag{15}$$

where $v_{0,0}(u)=\int_u^{\infty}\int_0^{\infty}p(x,y|0)dxdy=\alpha\int_u^{\infty}h(y)dy$. To obtain another explicit expression satisfied by $\psi(u)$, we take the Laplace transform of Eq. (11), and get that $\hat{\bar{G}}(s)=\alpha/s-\hat{g}(s)/s$. Substituting Eq. (12) into this equation followed by simple algebraic manipulations, we obtain

$$\hat{\bar{G}}(s)=\alpha\hat{\bar{G}}(s)\hat{h}(s)+\alpha\frac{1-\hat{h}(s)}{s}, \tag{16}$$

By inverting the Laplace transform of the equation (16), it follows that

$$\bar{G}(u)=\alpha\int_0^u\bar{G}(u-y)h(y)dy+\alpha\left(1-\int_0^u h(y)dy\right)=\alpha\int_0^u\bar{G}(u-y)h(y)dy+v_{0,0}(u),\quad u\geq 0. \tag{17}$$

By Eq. (3), we have

$$\phi_{0,0}(u)=\alpha\int_0^u \phi_{0,0}(u-y)h(y)dy+\nu_{0,0}(u). \tag{18}$$

Noting that a defective renewal equation uniquely defines a function. Comparing Eqs. (17) and (18) yields

$$\psi(u)=\phi_{0,0}(u)=\bar{G}(u). \tag{19}$$

By (10) and (11), we have

$$\bar{G}(y)=\sum_{n=1}^{\infty}\left((1-\alpha)\alpha^n\int_y^{\infty}h^{*n}(x)dx\right),\quad y\geq 0, \tag{20}$$

From (19) and (20), it follows that

$$\psi(u)=\sum_{n=1}^{\infty}\left((1-\alpha)\alpha^n\int_u^{\infty}h^{*n}(y)dy\right),\quad u\geq 0. \tag{21}$$

In particular, $\psi(0)=\sum_{n=1}^{\infty}\left((1-\alpha)\alpha^n\int_0^{\infty}h^{*n}(y)dy\right)=\alpha$.

4. Numerical results

In this section, we consider numerical results of the ruin probability under the risk model (1) for exponentially distributed claim sizes.

Let $f(x)=\beta e^{-\beta x}$. Then

$$F(x)=1-e^{-\beta x},\quad \hat{f}(s)=\frac{\beta}{\beta+s},\quad f_x(y)=f(x+y)\big/\big(1-F(x)\big)=\beta e^{-\beta y},$$

$$h(y)=\frac{1}{\alpha}\int_0^{\infty}p(x,y|0)dx=\frac{1}{\alpha}\int_0^{\infty}f_{U_{T-}}(x|0)f_x(y)dx=\beta e^{-\beta y}. \tag{22}$$

It follows from (16) and (22) that

$$\hat{\bar{G}}(s)=\alpha\hat{\bar{G}}(s)\frac{\beta}{\beta+s}+\frac{\alpha}{\beta+s}=\frac{\alpha}{\beta(1-\alpha)+s},$$

By inverting the Laplace transform of the last equation, one easily deduces that

$$\psi(u)=\bar{G}(u)=\alpha e^{-\beta(1-\alpha)u}. \tag{23}$$

To determine α, we again use the cumulative law of mathematical expectation. By conditioning on the time T_1 and the amount X_1 of the first claim and considering when ruin occurs on the first claim, we also have

$$\phi_{0,0}(u)=\int_0^{\infty}\int_{u+ct}^{\infty}(u+ct)^0(y-u-ct)^0\beta e^{-\beta y}k(t)dydt+\int_0^{\infty}\int_0^{u+ct}\phi_{0,0}(u+ct-y)\beta e^{-\beta y}k(t)dydt.$$

By $\phi_{0,0}(u)=\bar{G}(u)$, the above equation can be written in the following form

$$\bar{G}(u)=\int_0^{\infty}\left(\int_0^{u+ct}\bar{G}(u+ct-y)\beta e^{-\beta y}dy+e^{-\beta(u+ct)}\right)k(t)dt. \tag{24}$$

From (17) and (22), we have

$$\bar{G}(u)=\alpha\int_0^u\bar{G}(u-y)\beta e^{-\beta y}dy+\alpha e^{-\beta u}, \tag{25}$$

By (25), it follows that

$$\int_0^{u+ct}\bar{G}(u+ct-y)\beta e^{-\beta y}dy+e^{-\beta(u+ct)}=\frac{1}{\alpha}\bar{G}(u+ct),$$

Substituting the above equation into Eq. (24) yields

$$\alpha\bar{G}(u)=\int_0^{\infty}\bar{G}(u+ct)k(t)dt\,, \tag{26}$$

It follows from (23) that

$$\bar{G}(u+ct)=\alpha e^{-\beta(1-\alpha)(u+ct)}=\bar{G}(u)e^{-\beta(1-\alpha)ct}\,, \tag{27}$$

Substituting Eq. (27) into Eq. (26) leads to

$$\alpha=\int_0^{\infty}e^{-\beta(1-\alpha)ct}k(t)dt=\hat{k}\left(c\beta(1-\alpha)\right).$$

In summary, the ruin probability is

$$\psi(u)=\alpha e^{-\beta(1-\alpha)u}\,, \tag{28}$$

where α is the unique real root in the interval $(0,1)$ to the equation

$$\xi=\hat{k}\left(c\beta(1-\xi)\right). \tag{29}$$

Here, see Section 3.1 of [8] for more details of a proof of the uniqueness.

In the following examples we will discuss the effect of the initial surplus u and the interclaim time T_i on the ruin probability $\psi(u)$.

Example 1. We assume the interclaim time $T_i\sim\Gamma(n,\lambda), i=1,2,3,\cdots$, with probability density function and the Laplace transform of it as follows:

$$k(t)=\frac{\lambda^n e^{-\lambda t}t^{n-1}}{(n-1)!},\quad \hat{k}(s)=\left(\frac{\lambda}{\lambda+s}\right)^n.$$

Suppose $c=1.5,\beta=0.8,\lambda=1$. In order to see the effect of the variation of the parameter n on the ruin probability $\psi(u)$, let n take the values 1, 2, and 3, respectively, and solve Eq. (29) to obtain the corresponding values of α are 0.8333, 0.2925, and 0.1138, respectively. From (28), we take advantage of the package *Mathematica 12.0* to examine the effect of the variation of n on $\psi(u)$ for $0\le u\le 10$, see Figures 1-3. From Figures 1-3, it is observed that $\psi(u)$ is a decreasing function of u and is a decreasing function of n.

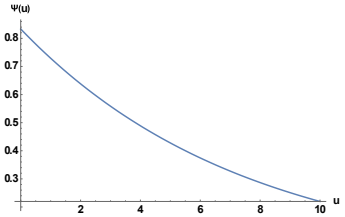

Figure 1. $\psi(u)$ **when** $n=1$.

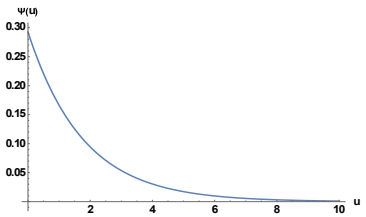

Figure 2. $\psi(u)$ **when** $n=2$.

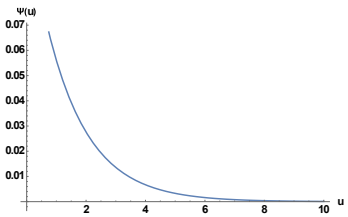

Figure 3. $\psi(u)$ **when** $n=3$.

Example 2. We assume the interclaim time $T_i=T_{i1}+T_{i2}, i=1,2,3,\cdots$, which is the sum of two independent random variables. $\{T_{ij}, i=1,2,\cdots\}$ is a sequence of non-negative, independent and identically distributed, random variables which are exponentially distributed with the parameter λ_j, $j=1,2$. We have the Laplace transform of the probability density function of T_i as follows:

$$\hat{k}(s)=\left(\frac{\lambda_1}{\lambda_1+s}\right)\left(\frac{\lambda_2}{\lambda_2+s}\right).$$

Suppose $c=1.5,\beta=0.8,\lambda_1=1$. In order to see the effect of the variation of the parameter λ_2 on the ruin probability $\psi(u)$, let λ_2 take the values 0.4, 0.6, and 0.8, respectively, and solve Eq. (29) to obtain the corresponding values of α

are 0.1368, 0.1948, and 0.2465, respectively. From (28), we take advantage of the package *Mathematica 12.0* to examine the effect of the variation of λ_2 on $\psi(u)$ for $0 \le u \le 10$, see Figures 4-6. From Figures 4-6, it is observed that $\psi(u)$ is a decreasing function of u and is an increasing function of λ_2.

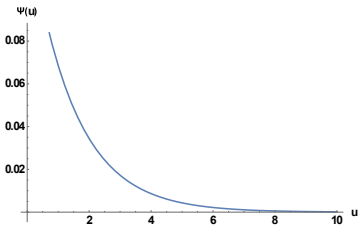

Figure 4. $\psi(u)$ **with** $\lambda_2 = 0.4$.

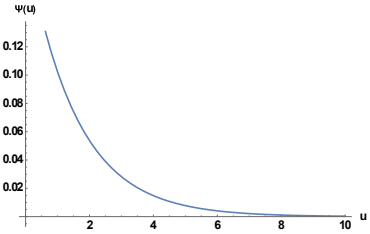

Figure 5. $\psi(u)$ **with** $\lambda_2 = 0.6$.

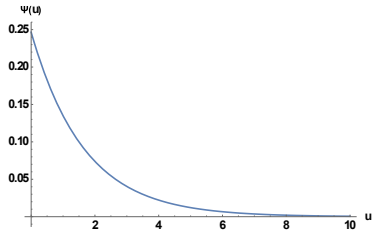

Figure 6. $\psi(u)$ **with** $\lambda_2 = 0.8$.

5. Conclusion

We propose a renewal risk model with an arbitrary interclaim time distribution and an arbitrary claim size distribution. A defective renewal equation satisfied by the joint moments of the surplus immediately before ruin and the deficit at ruin is given under this model. Solving this renewal equation yields an explicit expression for the above joint moments (including the ruin probability). In particular, an explicit expression for the ruin probability is obtained under the exponential claims condition. Numerical examples are given for the two cases where the claims counting process is an Erlang(n) process and a generalized Erlang(2) process, respectively, to analyze the effect of the relevant parameters on the ruin probability and to verify the validity of the above conclusions. The risk model proposed in this paper has wider applicability than the current mainstream risk model with interclaim times following a specific distribution. Numerical examples show that the explicit expressions of the joint moments in this paper have certain application value for insurance companies to correctly calculate the ruin risk and to understand their vulnerability in the ruin event.

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