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# On Expectation of Generating Certain Finite Nilpotent Groups

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## Abstract

Let $G$ be a finite group. Define $E(G)$ to be the expected number of elements of $G$ which have to be drawn at random with replacement from $G$ before a set of generators is found. The purpose of this paper is to compute $E(G)$ for certain nilpotent groups. This paper is divided into six sections. The first section is an introduction. In the second section we present the definition $\lambda_n(G)$ the probability that $n$ elements drawn at random, with replacement from a finite group $G$ generate $G$, and the definition $E(G)$ the expectation of the group $G$, that is the expected number of elements of $G$ which have to be drawn at random, with replacement, before a set of generators is found. In the third section we present some preliminaries and earlier results e.g if $G$ is a group and $\Phi(G)$ is its Frattini subgroup. Then, $\lambda_n(G) = \lambda_n(G/\Phi(G))$. In section four, we computed $E(G)$ for certain nilpotent groups such as $\mathbb{Z}_{p^4}$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. In section five, we computed $E(G)$ for a non-trivial example. The last section is a conclusion.

## Keywords

Probability, expectation, nilpotent group, $p$ −groups, Euler function

## 1. Introduction

In [1], V. Acciaro computed the probability of generating some common families of finite groups and the function $E(G)$ for some common classes of groups, starting from the $p$-groups and the groups which are the direct product of groups of coprime order and also analyzed the class of nilpotent groups. The expectation function $E(G)$ is of great importance in describing certain algorithms for testing whether a group $G$ generated by a given set of $m$ permutations of degree $n$ is regular $i.e$ if its order and degree are equal. A survey of algorithms and the expected execution time of these algorithms to handle permutation groups has been studied in [2]. In this paper, we compute $E(G)$ for certain nilpotent groups. For more information refer to [3, 5, 7].

## 2. Definition

The following definitions have been taken from [4].

**Definition 1.** *An ordered set $(x_1, \ldots, x_n)$ of $n$ elements of a group $G$, not necessarily distinct which generates $G$ is called an $n$-basis of $G$.*

**Definition 2.** *The $n-th$ Eulerian function $\phi_n(G)$ is the number of distinct $n$-basis of $G$.*

**Note 3.** *If the group $G$ cannot be generated by $n$ elements, then $\phi_n(G) = 0$ and if $G$ is cyclic of order $n$ then $\phi_1(G) = \phi(G)$, where $\phi(n)$ is the ordinary Eulerian function of an integer.*

**Note 4.** *Let $(g_1, \ldots, g_n)$ be an $n-$tuple of $G$. Then either $G = \langle g_1, \ldots, g_n \rangle$ or $H = \langle g_1, \ldots, g_n \rangle$ where $H$ is a subgroup of $G$. Hence one has the following fundamental identity*

$$|G|^n = \sum_{H \leq G} \phi_n(H).$$

**Definition 5.** *Let $\lambda_n(G)$ denote the probability that $n$ elements drawn at random, with replacement from $G$ generate $G$. Then one obtains*

$$\lambda_n(G) = \frac{\phi_n(G)}{|G|^n}.$$

**Definition 6.** *Let $E(G)$ denote the expected number of elements of $G$ which have to be drawn at random, with replacement, before a set of generators is found.*

The expectation function

$$E(G) = \sum_{d=1}^{\infty} d(\lambda_d(G) - \lambda_{d-1}(G))$$

where $\lambda_d(G) - \lambda_{d-1}(G)$ is the probability that $G = \langle g_1, \ldots, g_d \rangle$ and $G \neq \langle g_1, \ldots, g_{d-1} \rangle$.

## 3. Preliminaries and earlier results

**Note 7.** *The following basic combinatorial identity that we will use to simplify the computation of $E(G)$.*

$$\sum_{d=1}^{\infty} x^{\frac{d}{d-1}} = (\frac{x}{x-1})^2,$$

where $x$ is a real number greater than $1$.

**Lemma 8.**

1. If $G$ is a group and $\Phi(G)$ is its Frattini subgroup. Then

$$\lambda_n(G) = \lambda_n(G/\Phi(G)).$$

2. If $G$ is a $p$-group with minimal number of generators $d$, then

$$\lambda_n(P^n) = \lambda_n(G),$$

where $P$ is an elementary abelian group isomorphic to $G/\Phi(G)$ of onder $p^d$.

3. If $G$ and $H$ are two finite groups of coprime order, then

$$\lambda_d(G \times H) = \lambda_d(G).\lambda_d(H).$$

4. If $G$ is nilpotent group of order $p_1^{e_1}, \ldots, p_k^{e_k}$, then

$$\lambda_d(G) = \prod_{i=1}^{k} \lambda_d(G_{p_i}),$$

where $G_{p_i}$ is the $p_i$-Sylow subgroup of $G$.

**Proof.** See [3, problems 8.7, 8.26 and 8.22].

**Lemma 9.**

$i)$ If $G$ is a $p$-group with minimal number of generators $d$, then

(a) $\lambda_d(G) = \prod_{i=1}^{d} (1 - p^{-i})$

(b) If $k \leq 0$, the $\lambda_{d+k}(G) = \lambda_d(G) \prod_{i=1}^{k} \frac{p^i - p^{-d}}{p^i - 1}$

$ii)$ If $G$ is a cyclic group of order $n$, then

$$\lambda_d(G) = \prod_{p|n} (1 - \frac{1}{p^d})$$

$iii)$ If $G$ is a $p$-group of order $p^2$, then

$$\lambda_d(G) = \begin{cases} \frac{p^d - 1}{p^d} & \text{if } G \text{ is cyclic} \\ 1 - p^{-d+1} + p^{1-2d} - p^{-d} & \text{if } G \text{ is elementary abelian} \end{cases}$$

$iv$) If $G$ is a group of order $pq$ with $p, q$ primes and $q < p$, then

(a) $\lambda_d(G) = \begin{cases} (\frac{p^d-1}{p^d})(\frac{q^d-1}{q^d}) & \text{if} q \nmid p-1 \\ 1-\frac{1}{p^d}-p^{d-1}+\frac{p^{d-1}}{q^d} & \text{if} q|p-1 \end{cases}$

**Proof.** See [1].

**Note 7.** *The following basic combinatorial identity that we will use to simplify the computation of* $E(G)$.

$$\sum_{d=1}^{\infty} x^{\frac{d}{d-1}} = (\frac{x}{x-1})^2,$$

where $x$ is a real number greater than $1$.

**Lemma 8.**

1. If $G$ is a group and $\Phi(G)$ is its Frattini subgroup. Then

$$\lambda_n(G) = \lambda_n(G/\Phi(G)).$$

2. If $G$ is a $p$-group with minimal number of generators $d$, then

$$\lambda_n(P^n) = \lambda_n(G),$$

where $P$ is an elementary abelian group isomorphic to $G/\Phi(G)$ of onder $p^d$.

3. If $G$ and $H$ are two finite groups of coprime order, then

$$\lambda_d(G \times H) = \lambda_d(G).\lambda_d(H).$$

4. If $G$ is nilpotent group of order $p_1^{e_1}, \ldots, p_k^{e_k}$, then

$$\lambda_d(G) = \prod_{i=1}^{k} \lambda_d(G_{p_i}),$$

where $G_{p_i}$ is the $p_i$-Sylow subgroup of $G$.

**Proof.** See [[3], problems 8.7, 8.26 and 8.22].

**Lemma 9.**

$i$) If $G$ is a $p$-group with minimal number of generators $d$, then

(a) $\lambda_d(G) = \prod_{i=1}^{d} (1-p^{-i})$

(b) If $k \leq 0$, the $\lambda_{d+k}(G) = \lambda_d(G) \prod_{i=1}^{k} \frac{p^i-p^{-d}}{p^i-1}$

$ii$) If $G$ is a cyclic group of order $n$, then

$$\lambda_d(G) = \prod_{p|n} (1-\frac{1}{p^d})$$

$iii$) If $G$ is a $p$-group of order $p^2$, then

$$\lambda_d(G) = \begin{cases} \frac{p^d-1}{p^d} & \text{if } G \text{ is cyclic} \\ 1-p^{-d+1}+p^{1-2d}-p^{-d} & \text{if } G \text{ is elementary abelian} \end{cases}$$

$iv$) If $G$ is a group of order $pq$ with $p, q$ primes and $q < p$, then

(a) $\lambda_d(G) = \begin{cases} (\frac{p^d-1}{p^d})(\frac{q^d-1}{q^d}) & \text{if} q \nmid p-1 \\ 1-\frac{1}{p^d}-p^{d-1}+\frac{p^{d-1}}{q^d} & \text{if} q|p-1 \end{cases}$

**Proof.** See [1].

## 4. Main results

In this section, we compute $E(G)$ for certain abelian groups $G$ of order $p^4$ namely $G \cong \mathbb{Z}_{p^4}$, $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and these groups are nilpotent as they are abelian.

**Proposition 10.** *Let* $G \cong \mathbb{Z}_{p^4}$. *Then* $E(G) = \frac{p}{p-1}$.

**Proof.** By Lemma 9(ii), $\lambda_d(\mathbb{Z}_{p^4}) = \frac{p^d-1}{p^d}$ and hence by Definition 6

$$E(\mathbb{Z}_{p^4}) = \sum_{d=1}^{\infty} d(\lambda_d(G) - \lambda_{d-1}(G))$$

$$= \sum_{d=1}^{\infty} d(\frac{p^d-1}{p^d} - \frac{p^{d-1}-1}{p^{d-1}})$$

$$= \sum_{d=1}^{\infty} \frac{d}{p^{d-1}}(\frac{p^d-1}{p} - p^{d-1} + 1)$$

$$= \sum_{d=1}^{\infty} \frac{d}{p^{d-1}}(\frac{p-1}{p}) = \frac{p-1}{p}(\frac{p}{p-1})^2,$$

using the basic combinatorial identity in Note 7

$$= \frac{p}{p-1} \text{(Compare with 4.11)}$$

**Proposition 11.** *Let* $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$, *then* $E(G) = \frac{2p^2+p}{p^2-1}$.

**Proof.** By Lemma 9(iv), $\lambda_d(G) = 1 - p^{-d+1} + p^{1-2d} - p^{-d}$ and hence by Definition 6 one has:

$$E(G) = \sum_{d=1}^{\infty} d(1 - p^{-d+1} - p^{-d} + p^{1-2d}) - (1 - p^{-d+1} - p^{-d+1} + p^{3-2d})$$

$$= \sum_{d=1}^{\infty} dp^{-d}(p^2-1) + \sum_{d=1}^{\infty} dp^{1-2d}(1-p^2)$$

$$= p^{-1}(p^2-1)\sum \frac{d}{p^{d-1}} + p^{-1}(1-p^2)\sum_{d=1}^{\infty}(\frac{d}{p^{2d-2}})\frac{p(p^2-1)}{(p-1)^2} - \frac{p(p^2)}{p^2-1}$$

using the basic combinatorial identity in Note 7

$$= \frac{2p^2+p}{p^2-1}. \text{(Compare with 4.21)}$$

**Proposition 12.** *Let* $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, *then* $E(G) = \frac{2p^2+p}{p^2-1}$.

The group $G$ is a $p$-group with minimal number of generators equal to two, that is $G \cong \langle b^p, a\rangle$ or $G \cong \langle a^p, a^i b\rangle$, $i = 0.1\ldots.p-1$. Therefore, the behavior of these groups is the same as the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$ of order $p^2$. Hence by Proposition 11, the claim follows

**Remark 13.** *The group* $G_2 = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ *has the following subgroups:*

1. It has $p + l$ cyclic subgroups of order $p$ namely

$$\mathbb{Z}_p \cong \langle a^p, b^{ip}\rangle, i = 1,2,\ldots,p.$$

$$\mathbb{Z}_p \cong \langle b^p\rangle.$$

2. It has $p^2 + p$ cyclic subgroups of order $p^2$ namely:

$$\mathbb{Z}_p^2 \cong \langle ab^i\rangle, i = 1,2,\ldots,p^2.$$

$$\mathbb{Z}_p^2 \cong \langle a^p b^j\rangle, j = 1,\ldots,p-1.$$

$$\mathbb{Z}_p^2 \cong \langle b\rangle$$

and one elementary abelian group of order $p^2$ that is $\mathbb{Z}_p \times \mathbb{Z}_p = \langle a^p b^p\rangle$

3. It has $p+1$ subgroups of order $p^3$ of the form

$$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \cong \langle a^p, a^i b \rangle, i = 0,1,\ldots,p-1.$$

$$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \cong \langle b^p, a \rangle.$$

4. It has also two trivial subgroups.
For the above remark, see [3].

**Proposition 14.** *Let* $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p^2$, *then*

$$\lambda_d(G) = 1 - p^{-4d}[1 - (p+1)(p^d - 1) - (p^{2+d} - p^{d+1})(p^d - 1) - (p+2)(p^{2d} - p^{d+1} - p^d + p)].$$

**Proof.** By applying the identity $|G|^d = \sum_{H \leq G} \phi_d(G)$ in Note 4, Remark 13 and Lemma 9, one obtains

$$\phi_d(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}) = p^{4d} - 1 - (p+1)(p^d - 1) - (p^2 + p)(p^d)(p^d - 1) -$$
$$(p^{2d} - p^{d+1} - p^d + p) - (p+1)(p^{2d} - p^{d+1} - p^d + p)$$

which implies

$$\lambda_d(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}) = \frac{\phi_d(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2})}{p^{4d}}$$

and hence the claim.

**Theorem 15.** *Let* $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. *Then*

$$E(G) = (p^2 + p - 1 - \frac{1}{p^2} - \frac{1}{p^3})(\frac{p^4}{p^4 - 1})^2 + (3 - \frac{2}{p^2} + 2p + p^2)(\frac{p^2}{p^2 - 1})^2 + (\frac{5}{p^2} + \frac{2}{p} + 3 + 4p)(\frac{p^3}{p^3 - 1})^2$$

**Proof.** Apply Definition 6 and Proposition 14, one has

$$E(G) = \sum_{d=1}^{\infty} d[(1 - p^{-4d} - p^{-4d}(p+1)(p^d - 1) - p^{-4d}(p^{2+d} + p^{d+1})(p^d - 1)$$
$$-(p+2)(p^{2d} - p^{d+1} - p^d + p)p^{-4d}) - (1 - p^{-4d+4} - p^{-4d+4}(p+1)(p^{d-1} - 1)$$
$$-p^{-4d}(p^{2d} + p^{d+1})(p^d - 1) - (p+2)(p^{2d} - p^{d+1} - p^d + p)p^{-4d})]$$
$$= \sum_{d=1}^{\infty} dp^{-4d}(p^4 - 1) - \sum_{d=1}^{\infty} dp^{-3d}(p+1) + \sum_{d=1}^{\infty} d(p+1)p^{-4d} -$$
$$\sum_{d=1}^{\infty} dp^{-2d}(p^2 + p) + \sum_{d=1}^{\infty} dp^{-3d}(p^2 + p) - \sum_{d=1}^{\infty} d(p+2)p^{-2d} +$$
$$\sum_{d=1}^{\infty} d(p+2)p^{-3d+1} + \sum_{d=1}^{\infty} d(p+2)p^{-3d} - \sum_{d=1}^{\infty} d(p+2)p^{-4d+1} +$$
$$\sum_{d=1}^{\infty} d(p+1)p^{-3d+3} - \sum_{d=1}^{\infty} d(p+1)p^{-4d+4} + \sum_{d=1}^{\infty} d(p+1)p^{-2d+3} -$$
$$\sum_{d=1}^{\infty} d(p+1)p^{-4d+4} + \sum_{d=1}^{\infty} d(p+2)p^{-2d+2} - \sum_{d=1}^{\infty} d(p+2)p^{-3d+4} +$$
$$\sum_{d=1}^{\infty} d(p+2)p^{-2d+2} - \sum_{d=1}^{\infty} d(p+2)p^{-3d+4} - \sum_{d=1}^{\infty} d(p+2)p^{-3d+3} +$$
$$\sum_{d=1}^{\infty} d(p+2)p^{-4d+5} = A\sum_{d=1}^{\infty} dp^{-4d} + B\sum_{d=1}^{\infty} dp^{-2d} + C\sum_{d=1}^{\infty} dp^{-3d}$$

where

$$A = p^6 + p^5 - p^4 - p^2 - p$$
$$B = 3p^2 + 2p^3 + p^4 - 2$$
$$C = p^5 + 4p^4 + 3p^3 + 2p^2 + 5p + 2$$

Hence

$$E(G) = Ap^{-4}\sum_{d=1}^{\infty}\frac{d}{(p^4)^{d-1}} + Bp^{-2}\sum_{d=1}^{\infty}\frac{d}{(p^2)^{d-1}} + Cp^{-3}\sum_{d=1}^{\infty}\frac{d}{(p^3)^{d-1}}$$
$$= Ap^{-4}(\frac{p^4}{p^4-1})^2 + Bp^{-2}(\frac{p^2}{p^2-1})^2 + Cp^{-3}(\frac{p^3}{p^3-1})^2,$$

using the basic combinatorial identity in Note 7. Hence the claim follows.

**Remark 16.** *The abelian group* $\boldsymbol{G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p}$ *has* $\boldsymbol{3p+5}$ *subgroups and these are* $\boldsymbol{p+1}$ *cyclic groups* $\boldsymbol{\mathbb{Z}_p}$ *of order* $\boldsymbol{p}$*. There are* $\boldsymbol{p+1}$ $\boldsymbol{p}$*-subgroups* $\boldsymbol{\mathbb{Z}_{p^2}}$ *or* $\boldsymbol{\mathbb{Z}_p \times \mathbb{Z}_p}$*,* $\boldsymbol{p+1}$ *subgroups of type* $\boldsymbol{\mathbb{Z}_{p^3}}$ *or* $\boldsymbol{\mathbb{Z}_{p^2} \times \mathbb{Z}_p}$ *and the trivial two subgroups G itself and the identity* $\{\boldsymbol{e}\}$*. The computation* $\boldsymbol{E(G)}$ *is similar to that of of* $\boldsymbol{E(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2})}$ *in Theorem 15, as* $\boldsymbol{\lambda_d(H)}$ *for the subgroup* $\boldsymbol{H}$ *of* $\boldsymbol{G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p}$ *is similar to that of* $\boldsymbol{G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}}$*, except the subgroup* $\boldsymbol{\mathbb{Z}_{p^3} \times \mathbb{Z}_p}$ *which is a* $\boldsymbol{p}$*-group with minimal number of generators is* $\boldsymbol{3}$*, hence its behavior is the same as* $\boldsymbol{\mathbb{Z}_{p^3}}$ *[by Lemma 8 (ii), (iii)].*

## 5. A non-trivial example

We conclude this paper with the computation of $E(G)$, where $G$ is a nilpotent group, $G \cong G_{p_1} \times G_{p_2} \times G_{p_3}$, $G_{p_i}$ is a $p_i$-Sylow subgroup of $G$. So

$$\lambda_d(G) = \prod_{i=1}^{3}\lambda_d(G_{p_i}) = \prod_{i=1}^{3}(1-\frac{1}{p_i^d}) = 1 - \sum_{d=1}^{3}\frac{1}{p_i^d} + \sum_{1\le i<j\le 3}\frac{1}{p_i^d p_j^d} - \frac{1}{p_1^d p_2^d p_3^d}$$

$$E(G) = \sum_{d=1}^{\infty} d(\lambda_d(G) - \lambda_{d-1}(G))$$

$$= \sum_{d=1}^{\infty} d((1 - \sum_{d=1}^{3}\frac{1}{p_i^d} + \sum_{1\le i<j\le 3}\frac{1}{p_i^d p_j^d} - \frac{1}{p_1^d p_2^d p_3^d}) - (1 - \sum_{d=1}^{3}\frac{1}{p_i^{d-1}} + \sum_{1\le i<j\le 3}\frac{1}{p_i^{d-1} p_j^{d-1}} - \frac{1}{p_1^{d-1} p_2^{d-1} p_3^{d-1}}))$$

$$= \sum_{i=1}^{3}(\sum_{d=1}^{\infty}\frac{d}{p_i^{d-1}}(1-\frac{1}{p_i})) - \sum_{1\le i<j\le 3}(\sum_{d=1}^{\infty}\frac{d}{p_i^{d-1}p_j^{d-1}}(1-\frac{1}{p_i p_j})) + (\sum_{d=1}^{\infty}\frac{d}{p_1^{d-1}p_2^{d-1}p_3^{d-1}}(1-\frac{1}{p_1p_2p_3})$$

$$= \sum_{i=1}^{3}(1-\frac{1}{p_i})(\frac{p_i}{p_i-1})^2 - \sum_{1\le i<j\le 3}(1-\frac{1}{p_i p_j})(\frac{p_i p_j}{1-p_i p_j})^2 + (1-\frac{1}{p_1p_2p_3})(\frac{p_1p_2p_3}{1-p_1p_2p_3})^2.$$

So, if $|G| = 30$, then $G = G_2 \times G_3 \times G_5$ and

$$E(G) = (1-\frac{1}{2})(4)^2 + (1-\frac{1}{3})(\frac{3}{2})^2 + (1-\frac{1}{5})(\frac{5}{4})^2 - (1-\frac{1}{6})(\frac{6}{5})^2 - (1-\frac{1}{10})(\frac{10}{9})^2 - (1-\frac{1}{15})(\frac{15}{14})^2 + (1-\frac{1}{30})(\frac{30}{29})^2$$
$$\simeq 1.470.$$

## 6. Conclusion

This study shows that the methodology we have used to compute $E(G)$ is mainly, based on the the work of Acciaro, Eulerian function $\phi(G)$ and the basic combinatorial identity

$$\sum_{d=1}^{\infty}\frac{d}{x^{d-1}} = (\frac{x}{x-1})^2$$

Also, this study shows that the computations $E(G)$ for nilpotent groups requires to have good information about the

subgroup constructions of these groups in order to compute their Eulerian functions and then their $\lambda_n(G)$ and $E(G)$.

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