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Error Approximation of the Time Dependent Hyperbolic Differential Equation by Using the DG Finite Element Method

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Abstract

The paper offers a mathematical study to determine the error approximation of the numerical solution by applying the discontinuous Galerkin (DG) finite element method of the time dependent hyperbolic differential equation. The DG method is a dynamic numerical method with much mass compensation and more flexible meshing than other methods. This study is specified a general introduction and discuss about the discontinuous Galerkin Method for the time dependent hyperbolic differential equation. The hyperbolic problem satisfies the condition of the existence and uniqueness of DG solution. The error analysis of this problem is also established. It is a different and straightforward approach to the weak formulation to seek error analysis from all other finite element scheme which is given in the literature. The main goal of this study is to theoretically explore the convergence of the solution as well as to regulate the error approximation of the methods and show the validity of the results.

Keywords

Time Dependent, Hyperbolic equation, Discontinuous Galerkin, Finite element method

1. Introduction

This paper provides a theoretical observation to approximate the error of the solutions of the time dependent hyperbolic differential equation. The method is well suited for large scale time-dependent computations in which high accuracy is required. The discontinuous Galerkin (DG) method has been extensively studied and applied to a wide range of parabolic problems. Chi-Wang Shu analysed the discontinuous Galerkin finite element method (DGFEM) for the distributed first-order linear hyperbolic problems [1]. It derived an error estimator on general finite element meshes that are sharp in the mesh. Chunguang Xiong and Yuan Li represented the convergence properties of the DGFEM approximation of optimal control problem governed by convection-diffusion equations [2]. It exposed a posteriori error estimates and a priori error estimates for both the states, ad-joint, and the control variable approximation. For the optimal control problem, these estimates are apparently not available in the literature. In the book Beatrice Riviere [3], Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations; Theory and Implementation, provided a comprehensive introduction to Discontinuous Galerkin Methods for solving DG problems. It covered both the theoretical foundations of the method and its practical implementation. The developments of discontinuous Galerkin methods are established by [4-7]. They provided an overview of the development of DG methods and their applications to various partial differential equations. The book by Vit Dolejsi, and Miloslav Feistauer [8], is the mathematical theory of the discontinuous Galerkin method (DGM), which is a relatively new technique for the numerical solution of partial differential equations. The book is concerned with the DGM developed for differential equations and its applications to the numerical simulation of compressible flow. It deals with the theoretical as well as practical aspects of the DGM and treats the basic concepts and ideas of the DGM, as well as the latest significant findings and achievements in this area. Emmanuil H. Georgoulis represented the introduction to the finite element method (FEM) and discontinuous Galerkin methods for the numerical solution to partial differential equations [9]. Brezzi, F. and Marini, L.D. discussed the application of the DG method to hyperbolic problems, such as the wave equation [5]. It presented a framework for the analysis of the method's stability and convergence and highlights the advantages of DG over other methods for these types of problems. Hossain MS, Xiong C, and Sun H showed a priori as well as a posteriori error estimates for the first-order hyperbolic equation [10]. Hossain, M.S. and Xiong, C. used a different form of penalty parameter to establish the error analysis of the convection equation [11].

2. Problem formulation

Let Ω be a bounded polynomial domain in $\mathbb{R}^d, d = 1, 2\ or\ 3$, let $(0, T)$ be a time interval. For $f \in L^2\left(0, T; L^2(\Omega)\right), g_D$ in $L^2\left(0, T; H^{\frac{1}{2}}(\partial\Omega)\right)$, and $z_0 \in L^2(\Omega)$ we consider the hyperbolic problem with Dirichlet boundary condition:

$$\frac{\partial^2 u}{\partial t^2} - \nabla.(\beta\nabla u) + bu = f\ in\ (0, T) \times \Omega \tag{1}$$

$$z = g_D\ on\ (0, T) \times \Omega \tag{2}$$

$$z = g_0\ on\ \{0\} \times \Omega \tag{3}$$

The function $u(x, t)$ measures the deflection of the string and β denote a constant non zero velocity vector. A strong solution of the hyperbolic problem belongs to $C^\infty([0, T] \times \Omega)$ and satisfies $(1) - (3)$ pointwisely. A weak solution of the hyperbolic problem belongs to the space $L^2\left(0, T; H^1(\partial\Omega)\right) \cap H^1\left(0, T; L^2(\Omega)\right)$ and satisfies the variational formulation

$$\forall t > 0, \qquad \forall v \in H_0^1(\Omega), \qquad \left(\frac{\partial^2 u}{\partial t^2}, v\right)_\Omega + (\beta\nabla u, \nabla v)_\Omega = (f, v)_\Omega$$

$$\forall v \in H_0^1(\Omega), \qquad (z(0), v)_\Omega = (z_0, v)_\Omega$$

Let us now define a semidiscrete solution of the hyperbolic problem.

3. Semidiscrete formulation

We approximate the solution $u(t)$ by a function $U_h(t)$ that belongs to the finite dimensional space $D_k(\varepsilon_h)$ for all $t \geq 0$. The solution U_h is referred to as the semidiscrete solution, or sometimes as the continuous in time solution.

Let $v \in H^s(\varepsilon_h)$ for $s > \frac{3}{2}$, multiply (1) by v, integrate over one mesh element, use Green's theorem, and sum over all elements to obtain

$$\forall t > 0, \int_\Omega \frac{\partial^2 u}{\partial t^2} v + \sum_{E \in \varepsilon_h} \int_E \beta\nabla u.\nabla v - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\beta\nabla u.\boldsymbol{n}_e\}[v] + \epsilon \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\beta\nabla u.\boldsymbol{n}_e\}[u]$$
$$+ \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma_e^0}{|e|^{\gamma_0}} \int_e [u][v] = L(t; v), \qquad \dots\dots\dots\dots (4)$$

where

$$L(t; v) = \int_\Omega f(t)v + \sum_{e \in \partial\Omega} \int_e g_D(t)\left(\epsilon(\beta\nabla u.\boldsymbol{n}_e) + \frac{\sigma_e^0}{|e|^{\gamma_0}} v\right).$$

We define the energy norm for the hyperbolic problem

$$\|v\|_\varepsilon = \left(\sum_E \|\beta\nabla v\|_{L^2(e)}^2 + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma_e^0}{|e|^{\gamma_0}} \|v\|_{L^2(e)}^2\right)^{\frac{1}{2}}$$

We still denote the bilinear form by a_ϵ as

$$a_\epsilon(w, v) = \sum_{E \in \varepsilon_h} \int_E \beta\nabla w.\nabla v - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\beta\nabla w.\boldsymbol{n}_e\}[v] + \epsilon \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\beta\nabla u.\boldsymbol{n}_e\}[w] + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma_e^0}{|e|^{\gamma_0}} \int_e [w][v]$$

And we assume that coercivity of a_ϵ holds true for some $\kappa > 0$.

$$\forall v \in D_k(\varepsilon_h), \qquad \kappa\|v\|_\varepsilon^2 \le a_\epsilon(v,v). \tag{5}$$

Thus, the semidiscrete variational formulation is as follows: For all ≥ 0 , find $U_h(t) \in D_k(\varepsilon_h)$ such that

$$\forall t \ge 0, \forall v \in D_k(\varepsilon_h), \left(\frac{\partial^2 u}{\partial t^2}, v\right)_\Omega + a_\epsilon(U_h(t), v) = L(t; v), \tag{6}$$

$$\forall v \in D_k(\varepsilon_h), \qquad (U_h(0), v)_\Omega = (\tilde{u}_0, v)_\Omega \tag{7}$$

The initial condition $\tilde{u}_0$ can be chosen to be u_0 if u_0 belongs to the discrete space $D_k(\varepsilon_h)$, or it can be chosen to be $\tilde{u}(0)$, where $\tilde{u}$ is an approximation of u to be specified later. Using the global basis functions defined by

$$\Phi_i^E = \begin{cases} \widehat{\Phi}^\circ F_E(x), x \in E \\ 0, x \notin E \end{cases}$$

We can expand the semidiscrete solution

$$\forall t \in (0,T), \forall x \in \Omega, \; U_h(t,x) = \sum_{E \in \varepsilon_h} \sum_{i=1} \xi_i^E(t) \Phi_i^E(x) \tag{8}$$

The degree of freedom ξ_i^E's are functions of time. Let N_{el} denote the number of elements in the mesh. We can rename the basis functions and the degree of freedom such that

$$\{\Phi_i^E : 1 \le i \le N_{loc}, E \in \varepsilon_h\} = \{\widetilde{\Phi}_j : 1 \le j \le N_{loc}N_{el}\},$$

$$\{\xi_i^E : 1 \le i \le N_{loc}, E \in \varepsilon_h\} = \{\tilde{\xi}_j : 1 \le j \le N_{loc}N_{el}\},$$

4. Stability analysis

We drive stability bounds for numerical solution. Choosing $v = U_h(t)$ in (6) and using the coercivity result (5), we have

$$\frac{1}{2}\frac{d^2}{dx^2}\|U_h\|_{L^2(\Omega)}^2 + \kappa\|U_h\|_\varepsilon^2 \le |L(t; U_h(t))|.$$

From Cauchy-Schwarz's inequality, the right-hand side is bounded by

$$|L(t; U_h(t))| \le \|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} + \sum_{e \in \partial\Omega}\left(\|\beta\nabla U_h(t).\boldsymbol{n}_e\|_{L^2(e)} + \frac{\sigma_e^0}{|e|^{\gamma_0}}\|U_h(t)\|_{L^2(e)}\right)\|g_D(t)\|_{L^2(e)}.$$

Next, we use the trace inequality

$$\forall v \in \mathbb{P}_k(E), \forall e \subset \partial E, \|\nabla v.\boldsymbol{n}\|_{L^2(e)} \le \widehat{C}_t|e|^{\frac{1}{2}}|E|^{\frac{-1}{2}}\|\nabla v\|_{L^2(E)}$$

And Young's inequality

$$\forall \epsilon > 0, \qquad \forall a, b \in \mathbb{R}, \qquad ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$$

And as usual, the constant C is independent of the mesh size h. The derivation of similar bounds is done several times, we get

$$|L(t; U_h(t))| \le \|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} + \frac{\kappa}{2}\|U_h\|_\varepsilon^2 + C\sum_{e \in \partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g_D(t)\|_{L^2(e)}^2 \tag{9}$$

Therefore, we obtain the intermediate result:

$$\frac{1}{2}\frac{d^2}{dx^2}\|U_h\|_{L^2(\Omega)}^2 + \frac{\kappa}{2}\|U_h\|_\varepsilon^2 \le \|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} + C\sum_{e \in \partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g_D(t)\|_{L^2(e)}^2 \tag{10}$$

We present two possible approaches for obtaining the final a priori bound. The first one is more standard and uses Gronwall's inequality. The second approach takes advantage of poincare´'s inequality.

Approach using Gronwall's inequality: We simply bound

$$\|f(t)\|_{L^2(\Omega)}\|U_h(t)\|_{L^2(\Omega)} \le \frac{1}{2}\|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|U_h(t)\|_{L^2(\Omega)}^2$$

Multiply the equation by 2, and integrate from 0 to t

$$\frac{d}{dt}\|U_h(t)\|_{L^2(\Omega)}^2 + \kappa\int_0^t \|U_h(s)\|_\varepsilon^2 \le \int_0^t \|f(s)\|_{L^2(\Omega)}^2 + \int_0^t \|U_h(s)\|_{L^2(\Omega)}^2 + \|U_h(0)\|_{L^2(\Omega)}^2 + C\sum_{e\in\partial\Omega}\frac{1}{|e|^{\gamma_0}}\int_0^t \|g_D(t)\|_{0,e}^2$$

Then, by the continuous Gronwall's inequality, we conclude that

$$\|U_h(t)\|_{L^2(\Omega)}^2 + \kappa\int_0^t \|U_h(s)\|_\varepsilon^2 \le C\left(\int_0^t \|f(s)\|_{L^2(\Omega)}^2 + \|U_h(0)\|_{L^2(\Omega)}^2 + \sum_{e\in\partial\Omega}\frac{1}{|e|^{\gamma_0}}\int_0^t \|g_D(t)\|_{0,e}^2\right) \quad (11)$$

The constant C grows exponentially in time. We observe that this approach is valid for all primal DG methods with zero penalties.

Approach using poincare´'s inequality: If we use

$$\forall v\in H^1(\varepsilon_h), \qquad \|v\|_{L^2(e)} \le C\left(|||\nabla v|||_{H^0(\varepsilon_h)}^2 + \sum_{e\in\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{\gamma_0}}\|[v]\|_{L^2(\Omega)}^2\right)$$

And Young's inequality to bound $\|U_h\|_{L^2(\Omega)}$, we have

$$\frac{1}{2}\frac{d^2}{dx^2}\|U_h\|_{L^2(\Omega)}^2 + \frac{\kappa}{2}\|U_h\|_\varepsilon^2 \le \frac{\kappa}{4}\|U_h\|_\varepsilon^2 + C\|f(t)\|_{L^2(\Omega)}^2 + C\sum_{e\in\partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g_D(t)\|_{L^2(e)}^2.$$

After multiply by 2 and integrating from 0 to t

$$\frac{d}{dt}\|U_h(t)\|_{L^2(\Omega)}^2 + \frac{\kappa}{2}\int_0^t \|U_h(s)\|_\varepsilon^2 \le \left\|\widetilde{U}_0\right\|_{L^2(\Omega)}^2 + C\int_0^t \|f(s)\|_{L^2(\Omega)}^2 + C\sum_{e\in\partial\Omega}\frac{1}{|e|^{\gamma_0}}\int_0^t \|g_D(s)\|_{L^2(e)}^2,$$

Which is the same inequality as (11) modulo some multiplicative constants. However, the constant C is independent of time. This approach is valid if the penalty value σ_e^0 is positive for all faces e. The final result is stated in the following lemma.

Lemma: Assume that $\gamma_0 \ge (d-1)^{-1}$. There exists a positive constant C independent of h such that

$$\|U_h(t)\|_{L^\infty(0,T;L^2(\Omega))}^2 + \int_0^T \|U_h\|_\varepsilon^2 \le C\left\|\widetilde{U}_0\right\|_{L^2(\Omega)}^2 + C\|f(s)\|_{L^2(0,T;L^2(\Omega))}^2 + C\sum_{e\in\partial\Omega}\frac{1}{|e|^{\gamma_0}}\|g_D(s)\|_{L^2(0,T;L^2(\Omega))}^2 \quad (12)$$

5. Error analysis

We drive error estimates for the numerical error $u-U_h$ in the $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\varepsilon_h))$ norms. We first define the hyperbolic projection $\tilde{u}$ of the exact solution u:

$$\forall t\ge 0, \qquad \forall v\in D_k(\varepsilon_h), \qquad a_\epsilon(u(t)-\tilde{u}(t),v) = 0, \quad (13)$$

From the analysis of hyperbolic problem described in Chapter 2, we know that if u belongs to $L^2(0,T;H^s(\varepsilon_h))$ for $s\ge\frac{3}{2}$, the following error estimate holds:

$$\forall t\ge 0, \qquad \|u(t)-\tilde{u}(t)\|_\varepsilon \le Ch^{\min(k+1,s)-1}\|u(t)\|_{H^s(\varepsilon_h)} \quad (14)$$

In addition, if Ω is convex, error estimates in L^2 norm are

$$\forall t\ge 0, \qquad \|u(t)-\tilde{u}(t)\|_{L^2(\Omega)} \le Ch^{\min(k+1,s)}\|u(t)\|_{H^s(\varepsilon_h)} \quad (15)$$

$$\forall t\ge 0, \qquad \|u(t)-\tilde{u}(t)\|_{L^2(\Omega)} \le Ch^{\min(k+1,s)-1}\|u(t)\|_{H^s(\varepsilon_h)} \quad (16)$$

Theorem 1: Assume that u belongs to $H^1(0,T;H^s(\varepsilon_h))$ and that u_0 belongs to $H^s(\varepsilon_h)$ for $s>\frac{3}{2}$. Assume that $\gamma_0(d-1)^{-1}\ge 1$. Assume that σ_e^0 is sufficiently large for all e. Then, there is a constant C independent of h such that

$$\left(\int_0^T \|u(t)-U_h(t)\|_\varepsilon^2\right)^{\frac{1}{2}} \le Ch^{\min(k+1,s)-1}\|u(t)\|_{H^1(0,T;H^s(\varepsilon_h))},$$

$$\|u(t)-U_h(t)\|_{L^\infty(L^2(\Omega))} \le Ch^{\min(k+1,s)-\delta}\|u(t)\|_{H^1(0,T;H^s(\varepsilon_h))},$$

Where $\delta = 0$, if $\gamma_0 \geq 3(d-1)^{-1}$, if the mesh consists only of triangles and tetrahedral, and if $g_D \in D_k(\varepsilon_h)$. Otherwise, $\delta = 1$.

Proof: Since the scheme is constant, we obtain the following orthogonality equation:

$$\forall t \geq 0, \qquad \forall v \in D_k(\varepsilon_h), \qquad \left(\frac{\partial^2 (U_h - u)}{\partial t^2}, v\right)_\Omega + a_\epsilon\Big(\big(U_h(t) - u(t)\big), v\Big) = 0.$$

Defining $\chi = U_h - \tilde{u}$, we have for all $t > 0$ and for all $v \in D_k(\varepsilon_h)$

$$\left(\frac{\partial^2 \chi}{\partial t^2}, v\right)_\Omega + a_\epsilon\left(\frac{\partial \chi(t)}{\partial t}, , v\right) = \left(\frac{\partial^2 (u - \tilde{u})}{\partial t^2}, v\right)_\Omega + a_\epsilon\Big(\big(u(t) - \tilde{u}(t)\big), v\Big). \tag{17}$$

Using the definition of the hyperbolic projection, we obtain

$$\left(\frac{\partial^2 \chi}{\partial t^2}, v\right)_\Omega + a_\epsilon\left(\frac{\partial \chi(t)}{\partial t}, v\right) = \left(\frac{\partial^2 (u - \tilde{u})}{\partial t^2}, v\right)_\Omega. \tag{18}$$

Choosing $v = \chi(t)$ and using the coercivity of a_ϵ and the definition of the hyperbolic projection,

$$\forall t > 0 \qquad \frac{1}{2}\frac{d^2}{dt^2}\|\chi(t)\|_{L^2(\Omega)}^2 + \kappa\|\chi(t)\|_\varepsilon^2 \leq \left(\frac{\partial^2 (u - \tilde{u})}{\partial t^2}, \chi(t)\right)_\Omega \tag{19}$$

As in the proof of the stability bound, we can use either Gronwall's inequality or poincare´'s inequality to obtain the final estimate. If the penalty parameter σ_e^0 are positive for all e, we can bound the right-hand side of the equation above as

$$\left(\frac{\partial^2 (u - \tilde{u})}{\partial t^2}, \chi(t)\right)_\Omega \leq \left\|\frac{\partial^2 (u - \tilde{u})}{\partial t^2}\right\|_{L^2(\Omega)} \|\chi(t)\|_{L^2(\Omega)} \leq \frac{\kappa}{2}\|\chi(t)\|_\varepsilon^2 + \frac{1}{2\kappa}\left\|\frac{\partial^2 (u - \tilde{u})}{\partial t^2}\right\|_{L^2(\Omega)}^2.$$

Therefore, using the error estimates satisfied by the hyperbolic projection, we obtain

$$\frac{1}{2}\frac{d^2}{dx^2}\|\chi(t)\|_{L^2(\Omega)}^2 + \frac{\kappa}{2}\|\chi(t)\|_\varepsilon^2 \leq Ch^{2\min(k+1,s)-2\delta}\left|\left|\left|\frac{\partial^2 u}{\partial t^2}\right|\right|\right|_{H^s(\varepsilon_h)}^2 \tag{20}$$

Under certain conditions given in

$$\|p - p_h\|_{L^2(\Omega)} \leq Ch^{\min(k+1,s)}\|p\|_{H^s(\varepsilon_h)} \text{ and}$$
$$\|p - p_h\|_{L^2(\Omega)} \leq Ch^{\min(k+1,s)-1}\|p\|_{H^s(\varepsilon_h)}.$$

Now δ is zero. Next, we multiply (20) by 2 and integrate from 0 to t:

$$\frac{d}{dt}\|\chi(t)\|_{L^2(\Omega)}^2 + \kappa\int_0^t \|\chi(\tau)\|_\varepsilon^2 \leq \|\chi(0)\|_{L^2(\Omega)}^2 + Ch^{2\min(k+1,s)-2\delta}\left|\left|\left|\frac{\partial^2 u}{\partial t^2}\right|\right|\right|_{L^2(0,T;H^s(\varepsilon_h))}^2.$$

We conclude by noting that $\chi(0) = 0$ and by using triangle inequalities in the L^2 norm

$$\|u(t) - U_h(t)\|_{L^2(\Omega)} \leq \|\chi(t)\|_{L^2(\Omega)} + \|u(t) - \tilde{u}(t)\|_{L^2(\Omega)}$$

The triangle inequalities in the energy norm

$$\left(\int_0^T \|u(t) - U_h(t)\|_\varepsilon^2\right)^{\frac{1}{2}} \leq \left(\int_0^T \|u(t) - \tilde{u}(t)\|_\varepsilon^2\right)^{\frac{1}{2}} + \left(\int_0^T \|\tilde{u}(t) - U_h(t)\|_\varepsilon^2\right)^{\frac{1}{2}},$$

And the error estimates satisfied by $\tilde{u}$.

Theorem 2: Let $\epsilon = -1$, there exist a constant C independent of h such that

$$\left\|\frac{\partial^2 (u - \tilde{u})}{\partial t^2}\right\|_{L^2(0,t;L^2(\Omega))} \leq Ch^{\min(k+1,s)}\|u\|_{H^1(0,T;H^s(\varepsilon_h))}.$$

Proof: In the error equation (4.18), we choose $v = \chi(t)$

$$\left\|\frac{\partial \chi}{\partial t}\right\|_{L^2(\Omega)}^2 + a_\epsilon\left(\frac{\partial \chi}{\partial t}, \chi(t)\right) = \left(\frac{\partial \chi}{\partial t}, \frac{\partial (u-\tilde{u})}{\partial t}\right)_\Omega$$

Thus, using the symmetry property of a_ϵ, we have

$$\left\|\frac{\partial \chi}{\partial t}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\frac{\partial}{\partial t}a_\epsilon\big(\chi(t), \chi(t)\big) = \left(\frac{\partial \chi}{\partial t}, \frac{\partial (u-\tilde{u})}{\partial t}\right)_\Omega \le \frac{1}{2}\left\|\frac{\partial \chi}{\partial t}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|\frac{\partial (u-\tilde{u})}{\partial t}\right\|_{L^2(\Omega)}^2.$$

Integrating from 0 to t and using the fact that $\chi(0) = 0,$ we obtain

$$\int_0^t \left\|\frac{\partial \chi}{\partial t}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}a_\epsilon\big(\chi(t), \chi(t)\big) \le \frac{1}{2}a_\epsilon\big(\chi(0), \chi(0)\big) + \frac{1}{2}\int_0^t \left\|\frac{\partial (u-\tilde{u})}{\partial t}\right\|_{L^2(\Omega)}^2 \le Ch^{2\min(k+1,s)}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^s(\varepsilon_h))}.$$

Using coercivity of a_ϵ and the triangle inequality, we have

$$\left\|\frac{\partial (u-U_h)}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \le \left\|\frac{\partial (u-\tilde{u})}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial \chi}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \le Ch^{\min(k+1,s)}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^s(\Omega))}.$$

This concludes the proof.

6. Conclusion

This paper investigated the error approximation of the numerical solution by applying the Discontinuous Galerkin finite element method for the hyperbolic differential equation. It considered discontinuous Galerkin finite element approximations of time dependent hyperbolic equation. This work studied the effect of finite element spaces on the norm properties of DG solutions. The technique used in this paper can also be extended to the higher-order time depending on the scheme to obtain the $L^2(\Omega)$. error estimate of the above method with the optimal order of convergence.

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