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For Arbitrary Angles: A Constructive Approach to Establishing a Consistent Trisection Point Using Compass and Straightedge

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Abstract

The impossibility of trisecting any angle with a compass and straightedge arises from the inability to trisect the corresponding arc using these tools. This study presents a construction method to identify a fixed trisection point for arbitrary angles. The construction involves drawing a special line segment, which is one-sixth of the chord's length and passes through the midpoint of the chord along with one endpoint. A perpendicular line is then drawn from the other endpoint, forming a predefined right-angled triangle. The length of the hypotenuse, plus one-third of the special line segment, becomes a variable segment (Segment 1). The process is iteratively repeated to construct a new right-angled triangle (Triangle 1). This method is further extended to create subsequent right-angled triangles (Triangle 2) by utilizing the hypotenuse of the previous triangle and one-third of a new variable segment (Segment 2). The procedure continues, generating a sequence of variable standard segments. By repeatedly drawing perpendicular lines intersecting the arc, the trisection points of the angle are established.

Keywords

Compass and straightedge trisection of arbitrary angles, Arc height, Chord, Adjacent side, Special line segment

1. Introduction

The construction method presented in this paper originates from the trisection points of a 180-degree angle. By drawing an arc height on the chord of a 180-degree angle and taking one-sixth of the chord as a special line segment, the intersection point of the perpendicular line drawn from the special line segment and the arc forms a right-angled triangle. The hypotenuse of this triangle is twice the length of the special line segment. It can be easily proven that this point is a trisection point. As the angle decreases below 180 degrees, the intersection point obtained using this method forms a right-angled triangle with a hypotenuse significantly smaller than twice the length of the special line segment. The reduction, however, is not proportional to the decrease in angle.

By taking one-third of the sum of the hypotenuse and the special line segment as a variable segment (Segment 1), repeating the construction to create a new right-angled triangle (Triangle 1), and continuing this process to generate subsequent triangles (Triangle 2), it is observed that the hypotenuse of Triangle 2 is approximately equal to twice the length of Segment 2, with precision reaching 7-10 significant figures. The sum of these two segments, divided by three, becomes the standard segment. Repeating this process for various right-angled triangles shows that not only can the lengths be equal to 15 significant figures (limited by calculator precision), but they also remain equal for triangles constructed at different angles.

While the trisection points obtained in this construction method coincide with those obtained by trisecting the arc directly, the method does not directly trisect the arc. Instead, it involves drawing a special line segment equal to one-sixth of the chord's length at the midpoint of the chord. A right-angled triangle is formed by drawing a perpendicular line from the other endpoint of the special line segment to the arc. The process continues by taking one-third of the sum of the hypotenuse and the special line segment as a variable segment (Segment 1) and repeating the construction until the standard segment is obtained. The construction process serves as an example of field extension theory, gradually reducing the difference between the hypotenuse of the predefined right-angled triangle and twice the length of the special line segment until they become equal. Through calculations and verifications for various obtuse, right, and acute angles, this construction method has been proven effective for trisecting any given angle. Examples of obtuse, right, and acute angles are provided for demonstration purposes [1].

2. Theoretical Proof

2.1 First Example

Given: $\angle AOB$ approximately 100^0 (obtuse angle)

$$\overrightarrow{OA} = \overrightarrow{OB} = \sqrt{10} = 3.16227766016838，\overrightarrow{AB} = 5.2，\overrightarrow{AH} = \overrightarrow{BH} = 2.6，$$

$$\overrightarrow{OH} = 1.8，\overrightarrow{HT} = 1.36227766016838$$

To Find: $\angle AOC = \frac{\angle AOB}{3}$

Construction:

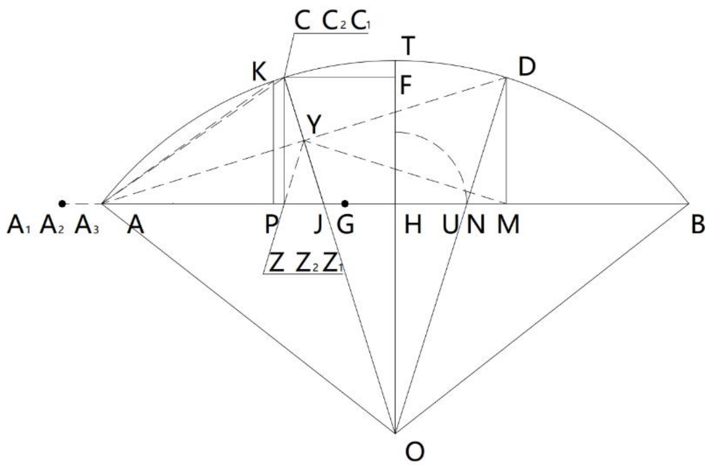

Figure 1. There exist fixed trisection points in any given angle.

1) Construct $\angle AOB$ approximately 100^0, with point O as the center, draw arc $\widehat{AB}$. Draw a perpendicular from point O to H ($\overrightarrow{OH} \perp \overrightarrow{AB}$), intersecting $\widehat{AB}$ at point T.
2) With point H as the center and $\frac{\overrightarrow{HT}}{2}$ as the radius, draw an arc intersecting $\overrightarrow{HB}$ at point U. Trisect $\overrightarrow{AU}$, where $\overrightarrow{GU} = \frac{\overrightarrow{AU}}{3} = \frac{\overrightarrow{TH}+\overrightarrow{AB}}{6}$.
3) With point H as the center and $\overrightarrow{GU}$ as the radius, draw an arc intersecting $\overrightarrow{AH}$ at point P. Draw $\overrightarrow{KP}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{KP} \perp \overrightarrow{AH}$), intersecting $\widehat{AB}$ at point K.
4) With P as the center, draw an arc with $\overrightarrow{AK}$ as the radius, intersecting the extension of $\overrightarrow{AB}$ at point A_1. Trisect $\overrightarrow{A_1H}$, where $\overrightarrow{A_1H}$, $\overrightarrow{Z_1H} = \frac{\overrightarrow{A_1H}}{3}$.
5) Draw $\overrightarrow{C_1Z_1}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C_1.
6) With Z_1 as the center and $\overrightarrow{AC_1}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_2. Trisect $\overrightarrow{A_2H}$, where $\overrightarrow{Z_2H} = \frac{\overrightarrow{A_2H}}{3}$.
7) Draw $\overrightarrow{C_2Z_2}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{C_2Z_2} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point Z_2.

8) With Z_2 as the center and $\overrightarrow{AC_2}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_3. Trisect $\overrightarrow{A_3H}$, where $\overrightarrow{ZH} = \frac{\overrightarrow{A_3H}}{3}$.
9) Draw $\overrightarrow{CZ}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{CZ} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C. Point C is the trisection point of $\angle AOB$.

[Note: The detailed steps for constructing the trisection point are provided based on the given parameters. This construction process serves as an illustration for the obtuse angle case.]

Proof: $\angle AOC = \frac{\angle AOB}{3}$

Auxiliary Lines:

Connect $\overrightarrow{OC}$ and intersect $\overrightarrow{AH}$ at point J.

Draw $\overrightarrow{CJ}$, find the midpoint Y of $\overrightarrow{CJ}$, and connect $\overrightarrow{YZ}$.

With Y as the center, draw an arc with $\overrightarrow{YA}$ as the radius, intersecting $\overrightarrow{HB}$ at point M.

From point M, draw $\overrightarrow{DM}$ perpendicular to $\overrightarrow{HB}$ ($\overrightarrow{DM} \perp \overrightarrow{HB}$), intersecting arc $\widehat{AB}$ at point D. Connect $\overrightarrow{YD}$. Connect $\overrightarrow{OD}$ and intersect $\overrightarrow{HB}$ at point N.

From point C, draw $\overrightarrow{CF}$ parallel to $\overrightarrow{AB}$ ($\overrightarrow{CF} \parallel \overrightarrow{AB}$), intersecting $\overrightarrow{OT}$ at point F [2].

Proof:

$$\because \overrightarrow{PH} = \frac{\overrightarrow{AH} + \frac{\overrightarrow{TH}}{2}}{3} = \frac{\overrightarrow{AB} + \overrightarrow{TH}}{6}, \quad \overrightarrow{AH} = 2.6, \quad \overrightarrow{TH} = \frac{1.36227766016838}{2} = 0.68113883008419,$$

$$\therefore \overrightarrow{PH} = \frac{5.2 + 1.36227766016838}{6} = \frac{2.6 + 0.68113883008419}{3} = 1.0937129433614$$

$$\because \overrightarrow{KP} + \overrightarrow{OH} = \sqrt{\overrightarrow{OK}^2 + \overrightarrow{PH}^2}, \ \overrightarrow{OK}^2 = 10, \quad \overrightarrow{PH}^2 = 1.19620767649457,$$

$$\therefore \overrightarrow{KP} + \overrightarrow{OH} = \sqrt{10 - 1.19620767649457} = \sqrt{8.80379232350543} = 2.9671185298483$$

$$\because \overrightarrow{KP} + \overrightarrow{OH} = 2.9671185298483, \quad \overrightarrow{OH} = 1.8$$

$$\therefore \overrightarrow{KP} = 2.9671185298483 - 1.8 = 1.1671185298483, \ \overrightarrow{KP}^2 = 1.36216566271526$$

$$\because \overrightarrow{AP} = \overrightarrow{AH} - \overrightarrow{PH}, \ \overrightarrow{AH} = 2.6, \ \overrightarrow{PH} = 1.0937129433614,$$

$$\therefore \overrightarrow{AP} = 2.6 - 1.09371279433614 = 1.50628720566386, \ \overrightarrow{AP}^2 = 2.26890114594664$$

$$\because \overrightarrow{AK} = \overrightarrow{AP}^2 + \overrightarrow{KP}^2, \quad \overrightarrow{KP}^2 = 1.36216566271526, \quad \overrightarrow{AP}^2 = 2.26890114594664,$$

$$\therefore \overrightarrow{AK} = \sqrt{1.36216566271526 + 2.26890114594664} = \sqrt{3.6310668086619} = 1.90553583242664$$

$$\because \overrightarrow{Z_1H} = \frac{\overrightarrow{AK} + \overrightarrow{PH}}{3}, \ \overrightarrow{AK} = 1.90553583242664, \ \overrightarrow{PH} = 1.0937129433614$$

$$\therefore \overrightarrow{Z_1H} = \frac{1.90553583242664 + 1.0937129433614}{3} = \frac{2.99924877578804}{3} = 0.999749591929347$$

$$\overrightarrow{Z_1H}^2 = 0.999749591929347^2 = 0.999499246562896,$$

$$\because \overrightarrow{C_1Z_1} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC}^2 - \overrightarrow{Z_1H}^2}, \ \overrightarrow{OC}^2 = 10, \quad \overrightarrow{Z_1H}^2 = 0.999499246562896,$$

$$\therefore \overrightarrow{C_1Z_1} + \overrightarrow{OH} = \sqrt{10 - 0.999499246562892} = \sqrt{9.0005007534371} = 3.00008345774532$$

$$\because \overrightarrow{C_1Z_1} + \overrightarrow{OH} = 3.00008345774532, \ \overrightarrow{OH} = 1.8$$

$$\therefore \overrightarrow{C_1Z_1} = 3.00008345774482 - 1.8 = 1.20008345774532, \ \overrightarrow{C_1Z_1}^2 = 1.44020030555396$$

$$\because \overrightarrow{AZ_1} = \overrightarrow{AH} - \overrightarrow{Z_1H}, \ \overrightarrow{AH} = 2.6, \quad \overrightarrow{Z_1H} = 0.999749591929347$$

$$\therefore \overrightarrow{AZ_1} = 2.6 - 0.999749591929347 = 1.600250408070651,$$

$$\overrightarrow{AZ_1}^2 = 1.600250408070651^2 = 2.56080136853028$$

$$\because \overrightarrow{AC_1} = \sqrt{\overrightarrow{AC_1}^2 + \overrightarrow{C_1Z_1}^2}, \quad \overrightarrow{AZ_1}^2 = 2.56080136853028, \quad \overrightarrow{C_1Z_1}^2 = 1.44020030555396$$

$$\therefore \overrightarrow{AC_1} = \sqrt{2.56080136853028 + 1.44020030555396} = \sqrt{4.00100167408424} = 2.00025040284566$$

$$\because \overrightarrow{Z_2H} = \frac{\overrightarrow{AC_1} + \overrightarrow{Z_1H}}{3}, \ \overrightarrow{AC_1} = 2.00025040284566, \ \overrightarrow{Z_1H} = 0.999749591929347$$

$\therefore \overrightarrow{Z_2H} = \frac{2.00025040284566 + 0.999749591929347}{3} = \frac{2.99999999477501}{3} = 0.999999998258337$

$\overrightarrow{Z_2H}^2 = 0.99999998258337^2 = 0.999999996516674$

$\because \overrightarrow{C_2Z_2} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC_2}^2 - \overrightarrow{Z_2H}^2}$，$\overrightarrow{OC_2}^2 = 10$，$\overrightarrow{Z_2H} = 0.999999996516674$，

$\therefore \overrightarrow{C_2Z_2} + \overrightarrow{OH} = \sqrt{10 - 0.999999996516674} = \sqrt{9.00000000348333} = 3.00000000058055$

$\because \overrightarrow{C_2Z_2} + \overrightarrow{OH} = 3.00000000058055$，$\overrightarrow{OH} = 1.8$，

$\therefore \overrightarrow{C_2Z_2} = 3.00000000058055 - 1.8 = 1.20000000058055$，$\overrightarrow{C_2Z_2}^2 = 1.44000000139332$

$\because \overrightarrow{AZ_2} = \overrightarrow{AH} - \overrightarrow{Z_2H}$，$\overrightarrow{AH} = 2.6$，$\overrightarrow{Z_2H} = 0.99999998258337$

$\therefore \overrightarrow{AZ_2} = 2.6 - 0.999999998258337 = 1.60000000174166$，$\overrightarrow{AZ_2}^2 = 2.56000000557331$

$\because \overrightarrow{AC_2} = \sqrt{\overrightarrow{C_2Z}^2 + \overrightarrow{AZ_2}^2}$，$\overrightarrow{C_2Z}^2 = 1.44000000139332$，$\overrightarrow{AZ_2}^2 = 2.56000000557331$

$\therefore \overrightarrow{AC_2} = \sqrt{1.44000000557331 + 2.56000000557331} = \sqrt{4.00000000696663}$

$\because \overrightarrow{ZH} = \frac{\overrightarrow{AC_2} + \overrightarrow{Z_2H}}{3}$，$\overrightarrow{AC_2} = 2.00000000174166$，$\overrightarrow{Z_2H} = 0.999999998258337 = 2.00000000174166$

$\therefore \overrightarrow{ZH} = \frac{2.00000000174166 + 0.999999998258337}{3} = \frac{3}{3} = 1$

$\because \overrightarrow{CZ} \perp \overrightarrow{ZH}$，$\overrightarrow{CF} \parallel \overrightarrow{AB}$，$\overrightarrow{ZH} = 1$

$\therefore \overrightarrow{CF} = \overrightarrow{ZH} = 1$，

$\because \overrightarrow{OF} = \sqrt{\overrightarrow{OC_2} - \overrightarrow{CF_2}}$，$\overrightarrow{OC_2} = 10$，$\overrightarrow{CF_2} = 1$，

$\therefore \overrightarrow{OF} = \sqrt{10 - 1} = \sqrt{9} = 3$，

$\therefore \overrightarrow{OF} = 3$，$\overrightarrow{OH} = 1.8$，$\frac{\overrightarrow{OF}}{\overrightarrow{OH}} = \frac{3}{1.8} = \frac{1}{0.6}$，$\frac{\overrightarrow{OF}}{\overrightarrow{OH}} = \frac{\overrightarrow{CF}}{\overrightarrow{JH}}$，$\overrightarrow{CF} = 1$

$\therefore \frac{\overrightarrow{CF}}{\overrightarrow{JH}} = \frac{1}{0.6}$，$\overrightarrow{JH} = 1 \times 0.6 = 0.6$，

$\because \overrightarrow{AJ} = \overrightarrow{AH} - \overrightarrow{JH}$，$\overrightarrow{AH} = 2.6$，$\overrightarrow{JH} = 0.6$，

$\therefore \overrightarrow{AJ} = \overrightarrow{AH} - \overrightarrow{JH} = 2.6 - 0.6 = 2$，

$\because \overrightarrow{CZ} \perp \overrightarrow{ZJ}$，$\overrightarrow{YC} = \overrightarrow{YJ}$，

$\therefore \overrightarrow{YZ} = \overrightarrow{YJ}$，$\angle YZJ = \angle YJZ$，

$\because \overrightarrow{YZ} = \overrightarrow{YJ}$，$\angle YZJ = \angle YJZ$，$\overrightarrow{YA} = \overrightarrow{YM}$，$\angle YAM = \angle YMA$，

$\therefore \overrightarrow{AJ} = \overrightarrow{ZM}$

$\because \overrightarrow{ZH} = \frac{\overrightarrow{AJ}}{2}$，$\overrightarrow{AJ} = \overrightarrow{ZM}$,

$\therefore \overrightarrow{ZH} = \frac{\overrightarrow{ZM}}{2}$，$\overrightarrow{ZH} = \overrightarrow{HM}$，

$\because \overrightarrow{DM} \perp \overrightarrow{HM}$，$\overrightarrow{ZH} = \overrightarrow{HM}$，

$\therefore \overrightarrow{CZ} = \overrightarrow{DM}$，$\overrightarrow{CJ} = \overrightarrow{DN}$，$\angle CJZ = \angle DNM$，

$\because \overrightarrow{YZ} = \overrightarrow{YJ} = \frac{\overrightarrow{CJ}}{2}$，$\overrightarrow{CJ} = \overrightarrow{DN}$，$\angle YZJ = \angle CJZ$，$\angle CJZ = \angle DNM$，

$\therefore \overrightarrow{YZ} = \frac{\overrightarrow{DN}}{2}$，$\angle YZJ = \angle DNM$，$\overrightarrow{YZ} \parallel \overrightarrow{DN}$

$\because \overrightarrow{YZ} = \frac{\overrightarrow{DN}}{2}$，$\overrightarrow{YZ} \parallel \overrightarrow{DN}$

$\therefore \overrightarrow{AZ} = \overrightarrow{ZN} = \dfrac{\overrightarrow{AN}}{2}$， $\overrightarrow{AY} = \overrightarrow{YD} = \dfrac{\overrightarrow{AD}}{2}$

$\because \overrightarrow{YA} = \overrightarrow{YD} = \dfrac{\overrightarrow{AD}}{2}$， $\overrightarrow{OA} = \overrightarrow{OD}$,

$\therefore \overrightarrow{OY} \perp \overrightarrow{AD}$， $\angle OAD = \angle ODA$， $\angle AOC = \angle COD$，

$\because \overrightarrow{AY} \perp \overrightarrow{CJ}$， $\overrightarrow{YC} = \overrightarrow{YJ}$，

$\therefore \overrightarrow{AJ} = \overrightarrow{AC}$， $\angle CAY = \angle YAJ$，

$\because \angle CAY = \angle CAD = \angle DAB$， $\overrightarrow{OA} = \overrightarrow{OC} = \overrightarrow{OD} = \overrightarrow{OB}$，

$\therefore \angle COD = \angle DOB$,

$\because \angle COD = \angle DOB$， $\angle AOC = \angle COD$，

$\therefore \angle AOC = \angle COD = \angle DOB = \dfrac{\angle AOB}{3}$

2.2 Second Example

Given: $\angle AOB$ = 90° (right angle)

$$\overrightarrow{OA} = \overrightarrow{OB} = \sqrt{14.9282032302755} = 3.86370330515627，\overrightarrow{AB} = 5.46410161513774$$

$$\overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{OH} = 2.73205080756888，\overrightarrow{HT} = 1.1316524975874，$$

To find: $\angle AOC = \frac{\angle AOB}{3}$

Drawing:

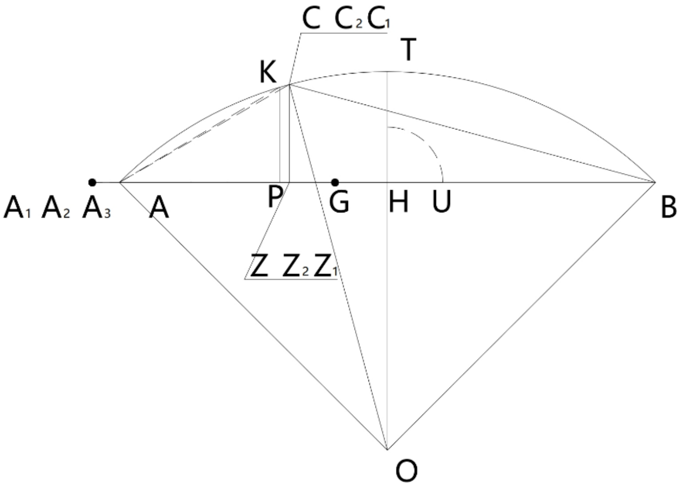

Figure 2. There exist fixed trisection points in any given angle.

Construct $\angle AOB = 90°$ with point O as the center, drawing arc $\widehat{AB}$. Draw $\overrightarrow{OH}$ perpendicular to $\overrightarrow{AB}$ ($\overrightarrow{OH} \perp \overrightarrow{AB}$), intersecting $\widehat{AB}$ at point T. Set $\overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{OH}$.

1) With point H as the center and $\frac{\overrightarrow{HT}}{2}$ as the radius, draw an arc intersecting $\overrightarrow{HB}$ at point U. Trisect $\overrightarrow{AU}$, where $\overrightarrow{AG} = \frac{2\overrightarrow{AU}}{3}$ and $\overrightarrow{GU} = \frac{\overrightarrow{AU}}{3} = \frac{\overrightarrow{TH}+\overrightarrow{AB}}{6}$.
2) With point H as the center and $\overrightarrow{GU}$ as the radius, draw an arc intersecting $\overrightarrow{AH}$ at point P. Draw $\overrightarrow{KP}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{KP} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point K.
3) With P as the center and $\overrightarrow{AK}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_1. Trisect $\overrightarrow{A_1H}$, where $\overrightarrow{Z_1H} = \frac{\overrightarrow{A_1H}}{3}$.
4) Draw a line from point Z_1, $\overrightarrow{C_1Z_1}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C_1.
5) With Z_1 as the center and $\overrightarrow{AC_1}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_2. Trisect

$\overrightarrow{A_2H}$, where $\overrightarrow{Z_2H} = \frac{\overrightarrow{A_2H}}{3}$.

6) Draw a line from point Z_2 , $\overrightarrow{C_2Z_2}$ perpendicular to AH ($C_2Z_2 \perp AH$), intersecting arc $\widehat{AB}$ at point C_2.
7) With Z_2 as the center and $\overrightarrow{AC_2}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_3. Trisect $\overrightarrow{A_3H}$, where $\overrightarrow{ZH} = \frac{\overrightarrow{A_3H}}{3}$.
8) Draw a line from point Z, $\overrightarrow{CZ}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{CZ} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C. Point C is the trisection point of $\angle AOB$. Connect $\overrightarrow{CB}$, $\angle AOC = \frac{\angle AOB}{3}$

Verification: $\angle AOC = \frac{\angle AOB}{3}$

Proof:

$\because \overrightarrow{PH} = \dfrac{\overrightarrow{AH} + \frac{\overrightarrow{HT}}{2}}{3} = \dfrac{\overrightarrow{AB} + \overrightarrow{TH}}{6}$, $\overrightarrow{AB} = 5.46410161513774$, $\overrightarrow{TH} = 1.1316524975874$

$\therefore \overrightarrow{PH} = \dfrac{5.46410161513774 + 1.1316524975874}{6} = \dfrac{6.59575411272541}{6} = 1.09929235212086$

$\overrightarrow{PH}^2 = 1.09929235212086^2 = 1.20844367543141$,

$\because \overrightarrow{KP} \perp \overrightarrow{PH}$, $\overrightarrow{KP} + \overrightarrow{OH} = \sqrt{\left(\overrightarrow{OK}^2 - \overrightarrow{PH}^2\right)}$, $\overrightarrow{OK}^2 = 14.9282032302755$, $\overrightarrow{PH} = \overrightarrow{PH}^2 = 1.20844367543141$,

$\therefore \overrightarrow{KP} + \overrightarrow{OH} = \sqrt{14.9282032302755 - 1.20844367543141} = \sqrt{13.7197595548441} = 3.70401937830299$

$\because \overrightarrow{KP} + \overrightarrow{OH} = 3.70401937830299, \overrightarrow{OH} = 2.73205080756888$

$\therefore \overrightarrow{KP} = 3.70401937830299 - 2.73205080756888 = 0.97196857073411$,

$\overrightarrow{KP}^2 = 0.944722902494909$

$\because \overrightarrow{AP} = \overrightarrow{AH} - \overrightarrow{PH}, \overrightarrow{AH} = 2.73205080756888$, $\overrightarrow{PH} = 1.09929235212086$

$\therefore \overrightarrow{AP} = 2.73205080756888 - 1.09929235212086 = 1.63275845544802, \overrightarrow{AP}^2 = 2.665900173837$

$\because \overrightarrow{AK} = \sqrt{\overrightarrow{AP}^2 + \overrightarrow{KP}^2}$, $\overrightarrow{KP}^2 = 2.665900173837$, $\overrightarrow{AP}^2 = 0.944722902494909$,

$\therefore \overrightarrow{AK} = \sqrt{2.665900173837 + 0.944722902494909} = \sqrt{3.61062307633191} = 1.90016396038129$

$\because \overrightarrow{Z_1H} = \frac{\overrightarrow{AK} + \overrightarrow{PH}}{3}$, $\overrightarrow{AK} = 1.90016396038129$, $\overrightarrow{PH} = 1.09929235212086$,

$\therefore \overrightarrow{Z_1H} = \dfrac{1.90016396038129 + 1.09929235212086}{3} = \dfrac{2.99945631250215}{3} = 0.99981877083405$

$\overrightarrow{Z_1H}^2 = 0.99981877083405^2 = 0.999637574512111$

$\because \overrightarrow{C_1Z_1} \perp \overrightarrow{Z_1H}, \overrightarrow{C_1Z_1} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC_1}^2 - \overrightarrow{Z_1H}^2}$,

$\overrightarrow{Z_1H}^2 = 0.999637574512111$, $\overrightarrow{OC_1}^2 = 14.9282032302755$,

$\therefore \overrightarrow{C_1Z_1} + \overrightarrow{OH} = \sqrt{14.9282032302755 - 0.999637574512111} = \sqrt{13.9285656557634}$

$\because \overrightarrow{C_1Z_1} + \overrightarrow{OH} = 3.73209936306141, \overrightarrow{OH} = 2.73205080756888 = 3.73209936306141$

$\therefore \overrightarrow{C_1Z_1} = 3.73209936306141 - 2.73205080756888 = 1.00004855549253$,

$\overrightarrow{C_1Z_1}^2 = 1.0000971133427$

$\because \overrightarrow{AZ_1} = \overrightarrow{AH} - \overrightarrow{Z_1H}$, $\overrightarrow{AH} = 2.73205080756888$, $\overrightarrow{Z_1H} = 0.99981877083405$

$\therefore \overrightarrow{AZ_1} = 2.73205080756888 - 0.99981877083405 = 1.73223203673483$,

$\overrightarrow{AZ_1}^2 = 3.0006278290905$

$\because \overrightarrow{AC_1} = \sqrt{\overrightarrow{AZ_1}^2 + \overrightarrow{C_1Z_1}^2}$, $\overrightarrow{AZ_1}^2 = 3.0006278290905$, $\overrightarrow{C_1Z_1}^2 = 1.0000971133427$

$\therefore \overrightarrow{AC_1} = \sqrt{3.0006278290905 + 1.0000971133427} = \sqrt{4.0007249424332} = 2.00018122739746$

$\because \overrightarrow{Z_2H} = \dfrac{\overrightarrow{AC_1} + \overrightarrow{Z_1H}}{3}$, $\overrightarrow{AC_1} = 2.00018122739746$,

$\overrightarrow{Z_1H} = 0.99981877083405$

$\therefore \overrightarrow{Z_2H} = \frac{2.00018122739746 + 0.99981877083405}{3} = \frac{2.99999999823151}{3} = 0.999999999410503$

$\overrightarrow{Z_2H}^2 = 0.999999999410503^2 = 0.99999999882101,$

$\because \overrightarrow{C_2Z_2} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC_2}^2 - \overrightarrow{Z_1H}^2}$, $\overrightarrow{OC_2}^2 = 14.9282032302755$,

$\overrightarrow{Z_2H}^2 = 0.99999999882101$

$\therefore \overrightarrow{C_2Z_2} + \overrightarrow{OH} = \sqrt{14.9282032302755 - 0.99999999882101} = \sqrt{13.9282032314545} = 3.73205080772683$

$\because \overrightarrow{C_2Z_2} + \overrightarrow{OH} = 3.73205080772683$, $\overrightarrow{OH} = 2.73205080756888$

$\therefore \overrightarrow{C_2Z_2} = 3.73205080772683 - 2.73205080756888 = 1.00000000015795$,

$\overrightarrow{C_2Z_2}^2 = 1.0000000003159$

$\because \overrightarrow{AZ_2} = \overrightarrow{AH} - \overrightarrow{Z_2H}$, $\overrightarrow{AH} = 2.73205080756888$, $\overrightarrow{Z_2H} = 0.999999999410503$

$\therefore \overrightarrow{AZ_2} = 2.73205080756888 - 0.999999999410503 = 1.73205080815838$,

$\overrightarrow{AZ_2}^2 = 3.0000000020421$

$\because \overrightarrow{AC_2} = \sqrt{\overrightarrow{C_2Z_2}^2 + \overrightarrow{AZ_2}^2}, \overrightarrow{C_2Z_2}^2 = 1.0000000003159, \overrightarrow{AZ_2}^2 = 3.0000000020421$

$\therefore \overrightarrow{AC_2} = \sqrt{1.0000000003159 + 3.0000000020421} = \sqrt{4.000000002358} = 2.0000000005895$,

$\because \overrightarrow{ZH} = \frac{\overrightarrow{AC_2} + \overrightarrow{Z_2H}}{3}$, $\overrightarrow{AC_2} = 2.0000000005895$, $\overrightarrow{Z_2H} = 0.999999999410503$

$\therefore \overrightarrow{ZH} = \frac{2.0000000005895 + 0.999999999410503}{3} = \frac{3}{3} = 1$

$\because \overrightarrow{CZ} \perp \overrightarrow{ZH}$, $\overrightarrow{CZ} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC}^2 - \overrightarrow{ZH}^2}$, $\overrightarrow{OC}^2 = 14.9282032302755$, $\overrightarrow{ZH}^2 = 1$

$\therefore \overrightarrow{CZ} + \overrightarrow{OH} = \sqrt{14.9282032302755 - 1} = \sqrt{13.9282032302755} = 3.73205080756888$

$\because \overrightarrow{CZ} + \overrightarrow{OH} = 3.73205080756888, \overrightarrow{OH} = 2.73205080756888$

$\therefore \overrightarrow{CZ} = 3.73205080756888 - 2.73205080756888 = 1$, $\overrightarrow{CZ}^2 = 1^2 = 1$

$\because \overrightarrow{ZB} = \overrightarrow{ZH} + \overrightarrow{HB}$, $\overrightarrow{ZH} = 1$, $\overrightarrow{HB} = 2.73205080756888$,

$\therefore \overrightarrow{ZB} = 1 + 2.73205080756888 = 3.73205080756888, \overrightarrow{ZB}^2 = 3.73205080756888^2 = 13.9282032302755$

$\because \overrightarrow{CZ} \perp \overrightarrow{ZB}$, $\overrightarrow{CB} = \overrightarrow{CZ}^2 + \overrightarrow{ZB}^2$, $\overrightarrow{CZ}^2 = 1$, $\overrightarrow{ZB}^2 = 13.9282032302755$

$\therefore \overrightarrow{CB} = \sqrt{1 + 13.9282032302755} = \sqrt{14.9282032302755} = 3.86370330515627$

$\because \overrightarrow{OC} = \overrightarrow{OB} = 3.86370330515627$, $\overrightarrow{CB} = 3.86370330515627$

$\therefore \overrightarrow{CB} = \overrightarrow{OC} = \overrightarrow{OB}$, $\angle COB = 60^\circ$

$\because \angle AOB = 90^0$, $\angle COB = 60^0$, $\angle COB = \frac{2\angle AOB}{3}$, $\angle AOC = \angle AOB - \angle COB$

$\therefore \angle AOC = \frac{\angle AOB}{3}$

2.3 Third Example

Given: $\angle A_1OB_1$ = 60° (acute angle)

$$\overrightarrow{OA} = \overrightarrow{OB} = \overrightarrow{AB} = \sqrt{33.1634374775264} = 5.75877048314364,$$

$\overrightarrow{AH} = 2.87938524157182$, $\overrightarrow{OH} = 4.98724153296638$, $\overrightarrow{HT} = 0.771528950177269$

To find: $\angle AOC = \frac{\angle AOB}{3}$

Drawing:

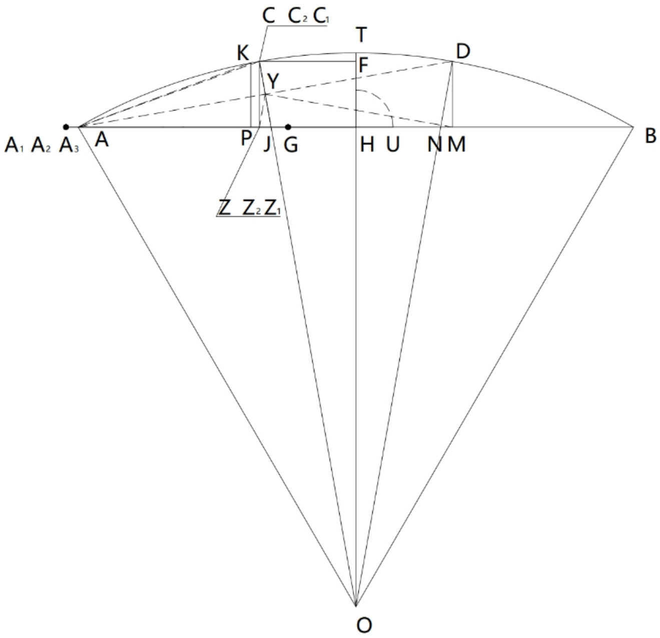

Figure 3. There exist fixed trisection points in any given angle.

Construct $\angle AOB$ approximately 100^0, with point O as the center, drawing arc $\widehat{AB}$. Draw $\overrightarrow{OH}$ perpendicular to $\overrightarrow{AB}$ ($\overrightarrow{OH} \perp \overrightarrow{AB}$), intersecting $\widehat{AB}$ at point T.

1) With point H as the center and $\frac{\overrightarrow{HT}}{2}$ as the radius, draw an arc intersecting $\overrightarrow{HB}$ at point U. Trisect $\overrightarrow{AU}$, where $\overrightarrow{GU} = \frac{\overrightarrow{AU}}{3} = \frac{\overrightarrow{TH}+\overrightarrow{AB}}{6}$.
2) With point H as the center and $\overrightarrow{GU}$ as the radius, draw an arc intersecting $\overrightarrow{AH}$ at point P. Draw $\overrightarrow{KP}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{KP} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point K.
3) With P as the center and $\overrightarrow{AK}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_1. Trisect $\overrightarrow{A_1H}$, where $\overrightarrow{Z_1H} = \frac{\overrightarrow{A_1H}}{3}$.
4) Draw a line from point Z_1, $\overrightarrow{C_1Z_1}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C_1.
5) With Z_1 as the center and $\overrightarrow{AC_1}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_2. Trisect $\overrightarrow{A_2H}$, where $\overrightarrow{Z_2H} = \frac{\overrightarrow{A_2H}}{3}$.
6) Draw a line from point Z_2, $\overrightarrow{C_2Z_2}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{C_2Z_2} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C_2.
7) With Z_2 as the center and $\overrightarrow{AC_2}$ as the radius, draw an arc intersecting the extension of $\overrightarrow{AB}$ at point A_3. Trisect $\overrightarrow{A_3H}$, where $\overrightarrow{ZH} = \frac{\overrightarrow{A_3H}}{3}$.
8) Draw a line from point Z, $\overrightarrow{CZ}$ perpendicular to $\overrightarrow{AH}$ ($\overrightarrow{CZ} \perp \overrightarrow{AH}$), intersecting arc $\widehat{AB}$ at point C. Point C is the trisection point of $\angle AOB$.

To prove: $\angle AOC = \frac{\angle AOB}{3}$

Auxiliary Lines: Connect $\overrightarrow{OC}$ and intersect $\overrightarrow{AH}$ at point J. Draw $\overrightarrow{CJ}$, find the midpoint Y of $\overrightarrow{CJ}$, and connect $\overrightarrow{YZ}$. With Y as the center, draw an arc with $\overrightarrow{YA}$ as the radius, intersecting $\overrightarrow{HB}$ at point M. From point M, draw $\overrightarrow{DM}$ perpendicular to $\overrightarrow{HB}$, intersecting arc $\widehat{AB}$ at point D. Connect $\overrightarrow{YD}$. Connect $\overrightarrow{OD}$ and intersect $\overrightarrow{HB}$ at point N. From point C, draw $\overrightarrow{CF}$ parallel to $\overrightarrow{AB}$, intersecting $\overrightarrow{OT}$ at point F.

Proof:

$$\because \overrightarrow{PH} = \frac{\overrightarrow{AH} + \frac{\overrightarrow{HT}}{2}}{3} = \frac{\overrightarrow{AB} + \overrightarrow{HT}}{6}, \ \overrightarrow{AB} = 5.75877048314364,$$

$\overrightarrow{HT} = 0.771528950177269,$

$\therefore \overrightarrow{PH} = \frac{\overrightarrow{AB} + \overrightarrow{HT}}{6} = \frac{5.75877048314364 + 0.771528950177269}{6} = \frac{6.53029943332091}{6} = 1.08838323888682$

$\because \overrightarrow{KP} \perp \overrightarrow{PH}, \overrightarrow{KP} + \overrightarrow{OH} = \sqrt{\overrightarrow{OK}^2 - \overrightarrow{PH}^2}$，$\overrightarrow{OK}^2 = 33.1634374775264$，$\overrightarrow{PH}^2 = 1.08838323888682^2 = 1.18457807468976$，

$\therefore \overrightarrow{KP} + \overrightarrow{OH} = \sqrt{33.1634374775264 - 1.18457807468976} = \sqrt{31.9788594028366} = 5.65498535832204$

$\because \overrightarrow{KP} + \overrightarrow{OH} = 5.65498535832204$，$\overrightarrow{OH} = 4.98724153296638$

$\therefore \overrightarrow{KP} = 5.65498535832204 - 4.98724153296638 = 0.66774382535566$，

$\overrightarrow{KP}^2 = 0.44588181630061$

$\because \overrightarrow{AP} = \overrightarrow{AH} - \overrightarrow{PH}$，$\overrightarrow{AH} = 2.87938524157182$，$\overrightarrow{PH} = 1.08838323888682$

$\therefore \overrightarrow{AP} = 2.87938524157182 - 1.08838323888682 = 1.791002002685$

$\overrightarrow{AP}^2 = 3.20768817362168$

$\because \overrightarrow{AK} = \sqrt{\overrightarrow{AP}^2 + \overrightarrow{KP}^2}$，$\overrightarrow{KP}^2 = 0.44588181630061$，$\overrightarrow{AP}^2 = 3.20768817362168$，

$\therefore \overrightarrow{AK} = \sqrt{0.44588181630061 + 3.20768817362168} = \sqrt{3.65356998992229} = 1.91143139817318$

$\because \overrightarrow{K_1G} = \overrightarrow{AG} - \overrightarrow{AK}$，$\overrightarrow{AG} = 2.17676647777364$，$\overrightarrow{AK} = 1.91143139817318$，

$\therefore \overrightarrow{K_1G} = 2.17676647777364 - 1.91143139817318 = 0.26533507960046$，

$\overrightarrow{K_1G_1} = \frac{\overrightarrow{K_1G}}{3} = 0.0884450265334865$，

$\because \overrightarrow{Z_1H} = \frac{\overrightarrow{AK} + \overrightarrow{PH}}{3}$，$\overrightarrow{AK} = 1.91143139817318$，$\overrightarrow{PH} = 1.08838323888682$，

$\therefore \overrightarrow{Z_1H} = \frac{1.91143139817318 + 1.08838323888682}{3} = 0.999938212353333$，

$\overrightarrow{Z_1H}^2 = 0.999876428524379$

$\because \overrightarrow{C_1Z_1} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC_1}^2 - \overrightarrow{Z_1H}^2}$，$\overrightarrow{OC_1}^2 = 33.1634374775264$，

$\overrightarrow{Z_1H}^2 = 0.999876428524379$

$\therefore \overrightarrow{C_1Z_1} + \overrightarrow{OH} = \sqrt{33.1634374775264 - 0.999876428524379} = \sqrt{32.163561049002}$
$= 5.67129271409985$

$\because \overrightarrow{C_1Z_1} + \overrightarrow{OH} = 5.67129271409985$，$\overrightarrow{OH} = 4.98724153296638$

$\therefore \overrightarrow{C_1Z_1} = 5.67129271409985 - 4.98724153296638 = 0.684051181133469$，

$\overrightarrow{C_1Z_1}^2 = 0.467926018410094$

$\because \overrightarrow{AZ_1} = \overrightarrow{AH} - \overrightarrow{Z_1H}$，$\overrightarrow{AH} = 2.87938524157182$，$\overrightarrow{Z_1H} = 0.999938212353335$

$\therefore \overrightarrow{AZ_1} = 2.87938524157182 - 0.999938212353335 = 1.87944702921848$，

$\overrightarrow{AZ_1}^2 = 3.53232113563817$

$\because \overrightarrow{AC_1} = \sqrt{\overrightarrow{AZ_1}^2 + \overrightarrow{C_1Z_1}^2}$，$\overrightarrow{AZ_1}^2 = 3.53232113563817$[3]，

$\overrightarrow{C_1Z_1}^2 = 0.467926018410094$

$\therefore \overrightarrow{AC_1} = \sqrt{3.53232113563817 + 0.467926018410094} = \sqrt{4.00024715404827} = 2.00006178755764$

$\because \overrightarrow{Z_2H} = \frac{\overrightarrow{AC_1} + \overrightarrow{Z_1H}}{3}$，$\overrightarrow{AC_1} = 2.00006178755764$，$\overrightarrow{Z_1H} = 0.999938212353335$

$\therefore \overrightarrow{Z_2H} = \frac{2.00006178755764 + 0.999938212353335}{3} = \frac{2.99999999991098}{3} = 0.999999999970327$

$\overrightarrow{Z_2H}^2 = 0.999999999970327^2 = 0.999999999940654$，

$\because \overrightarrow{C_2Z_2} \perp \overrightarrow{Z_2H}$， $\overrightarrow{C_2Z_2} + \overrightarrow{OH} = \sqrt{\overrightarrow{OC_2}^2 - \overrightarrow{Z_2H}^2}$， $\overrightarrow{OC_1}^2 = 33.1634374775264$，

$\overrightarrow{Z_2H}^2 = 0.999999999940654$,

$\therefore \overrightarrow{C_2Z_2} + \overrightarrow{OH} = \sqrt{33.1634374775264 - 0.999999999940654} = \sqrt{32.1634374775857} = 5.67128181962294$

$\because \overrightarrow{C_2Z_2} + \overrightarrow{OH} = 5.67128181962294$， $\overrightarrow{OH} = 4.98724153296638$

$\therefore \overrightarrow{C_2Z_2} = 5.67128181962294 - 4.98724153296638 = 0.684040286656559$，

$\overrightarrow{C_2Z_2}^2 = 0.467911113769187$

$\because \overrightarrow{AZ_2} = \overrightarrow{AH} - \overrightarrow{Z_2H}$， $\overrightarrow{AH} = 2.87938524157182$， $\overrightarrow{Z_2H} = 0.999999999970327$

$\therefore \overrightarrow{AZ_2} = 2.87938524157182 - 0.999999999970327 = 1.87938524160149$，

$\overrightarrow{AZ_2}^2 = 3.53208888634949$

$\because \overrightarrow{AC_2} = \sqrt{\overrightarrow{C_2Z_2}^2 + \overrightarrow{AZ_2}^2}$， $\overrightarrow{C_2Z_2}^2 = 0.467911113769187$， $\overrightarrow{AZ_2}^2 = 3.53208888634949$

$\therefore \overrightarrow{AC_2} = \sqrt{0.467911113769187 + 3.53208888634949} = \sqrt{4.00000000011868} = 2.00000000002967$

$\because \overrightarrow{ZH} = \dfrac{\overrightarrow{AC_2} + \overrightarrow{Z_2H}}{3}$， $\overrightarrow{AC_2} = 2.00000000002967$， $\overrightarrow{Z_2H} = 0.999999999970327$

$\therefore \overrightarrow{ZH} = \dfrac{2.00000000002967 + 0.999999999970327}{3} = \dfrac{3}{3} = 1$，

$\because \overrightarrow{CZ} \perp \overrightarrow{ZH}, \overrightarrow{CF} \parallel \overrightarrow{AB}$， $\overrightarrow{ZH} = 1$

$\therefore \overrightarrow{CF} = \overrightarrow{ZH} = 1$，

$\because \overrightarrow{OF} = \sqrt{\overrightarrow{OC}^2 - \overrightarrow{CF}^2}$， $\overrightarrow{OC}^2 = 33.1634374775264$， $\overrightarrow{CF}^2 = 1$，

$\therefore \overrightarrow{OF} = \sqrt{33.1634374775264 - 1} = \sqrt{32.1634374775264} = 5.67128181961771$

$\because \overrightarrow{OF} = 5.67128181961771$， $\overrightarrow{OH} = 4.98724153296638$，

$\therefore \dfrac{\overrightarrow{OF}}{\overrightarrow{OH}} = \dfrac{5.67128181961771}{4.98724153296638} = \dfrac{1}{0.879385241571818}$

$\because \dfrac{\overrightarrow{OF}}{\overrightarrow{OH}} = \dfrac{\overrightarrow{CF}}{\overrightarrow{JH}}$， $\dfrac{\overrightarrow{OF}}{\overrightarrow{OH}} = \dfrac{1}{0.879385241571818}$， $\dfrac{\overrightarrow{OF}}{\overrightarrow{OH}} = \dfrac{\overrightarrow{CF}}{\overrightarrow{JH}}$， $\overrightarrow{CF} = 1$

$\therefore \overrightarrow{JH} = 1 \times 0.879385241571818 = 0.879385241571818$

$\because \overrightarrow{AJ} = \overrightarrow{AH} -$ ， $\overrightarrow{AH} = 2.87938524157182$， $\overrightarrow{JH} = 0.879385241571818$，

$\therefore \overrightarrow{AJ} = \overrightarrow{AH} - \overrightarrow{JH} = 2.87938524157182 - 0.879385241571818 = 2$

$\because \overrightarrow{CZ} \perp \overrightarrow{ZJ}$， $\overrightarrow{YC} = \overrightarrow{YJ}$[4]，

$\therefore \overrightarrow{YZ} = \overrightarrow{YJ}$， $\angle YZJ = \angle YJZ$，

$\because \overrightarrow{YZ} = \overrightarrow{YJ}$， $\overrightarrow{YA} = \overrightarrow{YM}$， $\angle YZJ = \angle YJZ$， $\angle YAM = \angle YMA$

$\therefore \overrightarrow{AJ} = \overrightarrow{ZM}$

$\because \overrightarrow{ZH} = \dfrac{\overrightarrow{AJ}}{2}$， $\overrightarrow{AJ} = \overrightarrow{ZM}$，

$\therefore \overrightarrow{ZM} = 2\overrightarrow{ZH}$， $\overrightarrow{ZH} = \overrightarrow{HM}$，

$\because \overrightarrow{DM} \perp \overrightarrow{HM}$， $\overrightarrow{ZH} = \overrightarrow{HM}$，

$\therefore \overrightarrow{CZ} = \overrightarrow{DM}$， $\overrightarrow{CJ} = \overrightarrow{DN}$， $\angle CJZ = \angle DNM$，

$\because \overrightarrow{YZ} = \overrightarrow{YJ} = \dfrac{\overrightarrow{CJ}}{2}$， $\overrightarrow{CJ} = \overrightarrow{DN}$， $\angle YZJ = \angle CJZ = \angle DNM$，

$\therefore \overrightarrow{YZ} = \frac{\overrightarrow{DN}}{2}$，$\angle YZJ = \angle DNM$，$\overrightarrow{YZ} \parallel \overrightarrow{DN}$

$\because \overrightarrow{YZ} = \frac{\overrightarrow{DN}}{2}$，$\overrightarrow{YZ} \parallel \overrightarrow{DN}$

$\therefore \overrightarrow{AZ} = \overrightarrow{ZN} = \frac{\overrightarrow{AN}}{2}$，$\overrightarrow{AY} = \overrightarrow{YD} = \frac{\overrightarrow{AD}}{2}$

$\because \overrightarrow{YA} = \overrightarrow{YD}$，$\overrightarrow{OA} = \overrightarrow{OD}$，

$\therefore \overrightarrow{OY} \perp \overrightarrow{AD}$，$\angle AOC = \angle COD$

$\because \overrightarrow{AY} \perp \overrightarrow{CJ}$，$\overrightarrow{YC} = \overrightarrow{YJ}$，

$\therefore \overrightarrow{AJ} = \overrightarrow{AC}$，$\angle CAY = \angle YAJ$，

$\because \angle CAY = \angle CAD = \angle DAB$，$\overrightarrow{OA} = \overrightarrow{OC} = \overrightarrow{OD} = \overrightarrow{OB}$

$\therefore \angle COD = \angle DOB$，

$\because \angle COD = \angle DOB$，$\angle AOC = \angle COD$，

$\therefore \angle AOC = \angle COD = \angle DOB = \frac{\angle AOB}{3}$ [5]

3. Conclusion

The examples presented, representing obtuse, right, and acute angles, collectively demonstrate that a fixed trisection point for any angle can be constructed using a compass and straightedge. The construction method involves the following steps: At the midpoint of the chord, create a special segment equal to one-sixth of the sum of the arc height and the chord length. Draw a perpendicular line from the other endpoint of the chord, intersecting the arc and forming a predetermined right-angled triangle. Take one-third of the hypotenuse of this right-angled triangle, denoted as special variable segment 1. Repeat the construction method to generate new right-angled triangles (new right-angled triangle 1 and new right-angled triangle 2). Use the hypotenuse of the second right-angled triangle plus one-third of its special variable segment 2 to create a standard segment. Draw a perpendicular line from this standard segment to intersect the arc, determining the trisection point of the angle. This systematic approach provides a reliable and reproducible method for constructing a trisection point for any angle using a compass and straightedge. The fixed trisection point exists for any different angle, and the construction method presented here represents the discovery of the pattern for obtaining this point. The theoretical basis for the impossibility of a compass and straightedge trisection of any angle lies in the fact that a 20° angle cannot be constructed using these tools. In materials discussing the impossibility of compass and straightedge trisection, there is often a proof by contradiction that involves a counterexample: "Using proof by contradiction, let's assume that it is possible to trisect a 60° angle. If trisecting a 60° angle were possible, it would lead to the construction of a 20° angle. Therefore, if any angle could be trisected, it would imply the ability to extend the dimensionality of a finite field to 3 using compass and straightedge, which contradicts the conclusion that compass and straightedge constructions cannot achieve this. This contradiction demonstrates that trisecting any angle cannot be solved using a compass and straightedge." (Quoted from the Zhihu response by Kuang Shimin) The construction method includes a specific example featuring a 60° angle, where the adjacent side and chord of the constructed triangle are both equal to 5.75877048314364. The chord corresponding to the 20° angle, created through the construction, is measured at 2 units. Furthermore, the construction method demonstrates the creation of a line segment equal to the adjacent side in the context of a 90° angle. This evidence substantiates that the points generated by this construction method constitute fixed trisection points for angles of any magnitude. The construction method achieves this by employing a special segment to construct a predetermined right-angled triangle, where the hypotenuse gradually diminishes from being unequal to equal to twice the special segment. This systematic approach leads to the identification of trisection points for angles, providing proof of compass and straightedge trisection. The presented compass and straightedge construction process serves as a typical example of field extension, aligning with the conclusion that compass and straightedge constructions cannot achieve a dimensionality expansion to 3 in finite fields. This observation underscores that the construction method, despite expanding the field, does not contradict the established limitations on compass and straightedge constructions regarding the extension of dimensionality to 3 in finite fields. In essence, the key to trisecting any angle in this construction method lies in generating the chord corresponding to one-third of the arc. While the compass and straightedge cannot directly trisect the arc, the construction avoids declaring the impossibility of trisecting any angle. Instead of directly trisecting the arc, the method introduces a special segment, defined as one-sixth of the sum of the arc height and the chord. This special segment, following a systematic pattern, is gradually reduced to a standard segment. By constructing a line segment equal to twice the standard segment, the method establishes that the chord corresponds to one-third of the arc for any angle. This approach allows overcoming the challenge of impossibility without refuting the theory of impossibility itself.

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